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[illegible]. [illegible]“ * ”[illegible], [illegible].

[illegible]. [illegible], [illegible]. [illegible], [illegible], [illegible]. [illegible]. [illegible], n [illegible], [illegible] n [illegible] Jordan [illegible]. [illegible], [illegible].

[illegible], [illegible]; [illegible].

[illegible], [illegible], [illegible].

[illegible] �, [illegible].

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[illegible]

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[illegible],[illegible].

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§3 □□□□□ 13

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§4 □□□□□□,□□□□□□□□ 23

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§5 □□□ 28

§6 □□□□□,□□□□□□□□□□□ 32

§7 □□□□□□□□□□□ 38

§8 □□□□□□□□□□□□ 40

§9 □□□□□□ 48

§10 □□□□□ 56

§11 □□□ 67

□8□ □□□□ 72

§1 □□□□□□□ 72

§2 □□□□□□□□,□□□□□□ 82

§3 □□□□□□□ 92

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§4 □□□ 97

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§1 □□□□□□□□ 105

§2 □□□□□□□□ 112

§3 □□□□□□□□□ 115

§4 □□□□□□□□□□□□□ 122

§5 □□□□□□□□□□ 126

§6 Hamilton-Cayley□□ 133

§7 □□□□□□□□□□ 135

§8 □□□□□□□ 143

§ 9 [illegible] Jordan [illegible] …… 149

§ 10 [illegible] …… 156

[illegible] 10 [illegible] …… 162

§ 1 [illegible] …… 162

§ 2 [illegible] …… 170

§ 3 [illegible], [illegible] …… 176

§ 4 [illegible] …… 180

§ 5 [illegible] …… 183

*§ 6 [illegible] …… 188

[illegible] …… 193

[illegible] …… 220

[illegible] 7 [illegible]

[illegible], [illegible]. [illegible], [illegible]. [illegible], [illegible] n [illegible]. [illegible], [illegible].

[illegible]. [illegible].

§ 1 [illegible]

[illegible]:

$$-x^3+x^2+\frac{1}{2}, \quad [illegible]\ x\ [illegible]; \tag{1}$$

$$-i^3+i^2+\frac{1}{2}, \quad [illegible]\ i=\sqrt{-1}; \tag{2}$$

$$-A^3+A^2+\frac{1}{2}I, \quad [illegible]\ A=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{3}$$

[illegible], [illegible] (2) [illegible] (3) [illegible]

$$-i^3+i^2+\frac{1}{2}=i-\frac{1}{2}, \tag{4}$$

$$-A^3+A^2+\frac{1}{2}I=-A+\frac{3}{2}I. \tag{5}$$

[illegible], [illegible] i [illegible] A [illegible], [illegible]. [illegible] x [illegible], [illegible], [illegible], [illegible]:

[illegible] 1 [illegible] K [illegible], x [illegible] K [illegible]. [illegible] n, [illegible] K [illegible] $a_0, a_1, \ldots, a_n$, [illegible]

$$a_n x^n + a_{n-1}x^{n-1}+\ldots+a_1 x + a_0 \tag{6}$$

[illegible] K [illegible], [illegible]: [illegible], [illegible]. [illegible], [illegible] x [illegible].

□□□□□□□□□□□□□□□,□□ 0.

□□□□(6)□, $a_i x^i$ □□ **i** □□, $i = 1, \ldots, n$; a_0 □□□□□,□□□□□.

□□□ 1□□,□□ K□□□□□□□□□□□□□□□□□□□□□□□□□.

□□□□□ $f(x), g(x), \ldots$□ $f, g, \ldots$□□□□□□□□.

□ $f(x)$□□□□□(6).□□ a_n□ 0,□□ $a_n x^n$ □□□□ $f(x)$□□□; n □□ $f(x)$□□□,□□ $\deg f(x)$□ $\deg f$.

□□□□□□□□□□ $-$□,□□□□:

$$(-□)+(-□)=-□,$$

$$(-□)+n=-□,$$

$$-□<n,$$

□□ n□□□□□□□.

□□□□□□ a,□□ a□ K□ a□ 0.

□□□□□ K□□□□□□□□□□□□□□□□ $K[x]$.□ $K[x]$□□□□□□□□□□□□:

□

$$f(x)=□_{i=0}^{n} a_i x^i,\quad g(x)=□_{i=0}^{m} b_i x^i,$$

□□□ n□ m,□

$$f(x)+g(x) \overset{\text{def}}{=} □_{i=0}^{n} (a_i+b_i)x^i, \tag{7}$$

$$f(x)g(x) \overset{\text{def}}{=} □_{s=0}^{n+m} □_{i+j=s} a_i b_j \; x^s, \tag{8}$$

□ $f(x)+g(x)$□ $f(x)$□ $g(x)$□□,□ $f(x)g(x)$□ $f(x)$□ $g(x)$□□.

□□□□□□□□□□□□□□□□□□□□□□□□□□□: " f, g, h□ $K[x]$,□

1° □□□□□,□ $f+g=g+f$;

2° □□□□□,□ $(f+g)+h=f+(g+h)$;

3° □□□□□□□□:$0+f=f+0=f$;

4° □ $f(x)=□_{i=0}^{n} a_i x^i$,□□ $-f(x)=□_{i=0}^{n}(-a_i)x^i$,□

$$f+(-f)=(-f)+f=0,$$

□ $-f$ □ f □□□□;

5° □□□□□,□ $fg=gf$;

6° [illegible], [illegible] $(fg)h=f(gh)$;

7° [illegible] 1 [illegible]: $1f=f1=f$;

8° [illegible]:

$$f(g+h)=fg+fh,$$
$$(g+h)f=gf+hf.$$

[illegible]:

$$f-g \overset{\text{def}}{=} f+(-g). \tag{9}$$

[illegible] **1** [illegible] $f(x), g(x) \in K[x]$, [illegible]

$$\deg(f\pm g) \leqslant \max\{\deg f, \deg g\}; \tag{10}$$

$$\deg(fg)=\deg f+\deg g. \tag{11}$$

[illegible] $f=0$ [illegible] $g=0$, [illegible] (10), (11) [illegible]. [illegible] $f\neq 0$ [illegible] $g\neq 0$. [illegible]

$$f(x)=\sum_{i=0}^{n} a_i x^i, \quad g(x)=\sum_{i=0}^{m} b_i x^i,$$

[illegible] $a_n\neq 0$, $b_m\neq 0$. [illegible] $\deg f=n$, $\deg g=m$, [illegible] $n \geqslant m$ [illegible]

$$f(x)\pm g(x)=\sum_{i=0}^{n}(a_i\pm b_i)x^i.$$

[illegible] $$\deg(f\pm g)\leqslant n=\max\{\deg f, \deg g\}.$$

[illegible] $a_n b_m \neq 0$, [illegible] $a_n b_m x^{n+m}$ [illegible] $f(x)g(x)$ [illegible]. [illegible]

$$\deg(fg)=n+m=\deg f+\deg g.$$

[illegible] (1) [illegible], [illegible] $f\neq 0$ [illegible] $g\neq 0$, [illegible] $fg\neq 0$. [illegible], [illegible]

9° [illegible], [illegible] $fg=fh$ [illegible] $f\neq 0$, [illegible] $g=h$.

[illegible] $fg=fh$ [illegible], $f(g-h)=0$. [illegible] $f\neq 0$, [illegible] $g-h=0$. [illegible] $g=h$.

[illegible]:

[illegible] Z;

[illegible] K [illegible] $K[x]$;

[illegible] K [illegible] n [illegible] $M_n(K)$.

[illegible] Z, $K[x]$, $M_n(K)$ [illegible], [illegible], [illegible], [illegible], [illegible], [illegible], [illegible], [illegible].

[illegible]. [illegible].

[illegible] **2** [illegible] S [illegible], [illegible] $S\times S$ [illegible] S [illegible] S [illegible]

□□□.

□ Z, $K[x]$, $M_n(K)$ □□□□,□□□□□□□□□:

□□ 3 □ R□□□□□□□,□□□□□□□□□□,□□□□□□,□
□ $a+b$,□□□□□□□,□□ ab;□□□□□□□□□□□ 6□□□□□:
" a, b, c□ R, □

1° □□□□□,□ $(a+b)+c=a+(b+c)$;

2° □□□□□,□ $a+b=b+a$;

3° □ R□□□□ 0,□□ $a+0=a$,□ 0□ R□□□□;

4° □□ a,□ R□□□□ d,□□ $a+d=0$,□ d□ a□□□□,□□ $-a$;

5° □□□□□,□ $(ab)c=a(bc)$;

6° □□□□□□□□□□□□□,□

$$a(b+c)=ab+ac,$$
$$(b+c)a=ba+ca,$$

□□ R□□□□.

□□□□,□ R□□□□□□□□□;□□□□ a□□□□□□□□.

Z, $K[x]$, $M_n(K)$ □□□□□ 3□□□,□□□□□□□,□□□□□□□,□□ K□□□□□□□,□□ K□ n□□□□□.

□□□□□□ K□□□.

□ R□□□□□□□:

$$a-b \overset{\text{def}}{=} a+(-b). \tag{12}$$

□ R□□□□□□□□□□□□,□

$$ab=ba, \ \text{"}\ a, b\text{□}\ R,$$

□□ R□□□□.

□ R□□□□□□□□ e□□□□:

$$ea=ae=a, \ \text{"}\ a\text{□}\ R, \tag{13}$$

□□ e□ R□□□□(□).□□□ R□□□□□□□,□□□□□□□ R□,□□□□□□□□,□□□□□ 1.

□ R□□□□ a□□□□□□□□(□□□□),□□ R□□□□ b□0,□□ $ab=0$($ba=0$).□□□□□□□□□□□□□□□□.□□□□,0□□□□□□,□□□□□□.□ 0□□□□□□;□□□□□□□□□□□□□□□.

□□□ R□□□□□□□□□,□ R□□□□□□□.

□□□□ 1(□ 0)□□□□□□□□□□□□□.

Z, $K[x]$, K □□□□. $M_n(K)$□□□□.

□□□□ Z□,□□□□□□□□□□□□□□□□□□□□□□□□□□□,□□□,□□□□□□□:

□□ **4** □□□ R□□□□□□□ R_1 □□ R□□□□□□□□□□□□,□ R_1 □□ R□□□□□.

□□□4□□,□□ R_1 □□ R□□□□□,□ R□□□(□□)□□ R_1 □□□(□□).□□□□ 2□□,□□□□:

$$a, b \in R_1 \Longrightarrow a+b \in R_1, □ ab \in R_1.$$

□□□ R_1 □□ R□□□□□□□□.

□□□□:

□□ **2**[*] □ R□□□□□□□ R_1 □□□□□□□□□□□□□,R_1 □□ R□□□□□□□□□□,□

$$a, b \in R_1 \Longrightarrow a-b \in R_1,\ ab \in R_1.$$

x

K[x]□□□□□□□□□□□□□□□□□□□ S,□□□□□□□□□□□□□,□□ S□ K[x]□□□□□.□□ K[x]□□□□ 1□□ S,□□ 1□□ S□□□□.□□□□□□□□ K□ S□□□□□□□:□□□□ a□□□□□□□ a,□□ 0□□□□□□□.□□ σ□□□,□□ σ□□□□□□□□□,□

$$\sigma(a+b) = \sigma(a) + \sigma(b), \quad \forall a, b \in K;$$

$$\sigma(ab) = \sigma(a)\sigma(b), \quad \forall a, b \in K.$$

□□ $A \in M_n(K)$,□□□□□□□□□□□□ **K**□□□ A□□□□:

$$a_m A^m + a_{m-1} A^{m-1} + \dots + a_1 A + a_0 I,$$

□□ m□□□□□,$a_i \in K$, $i=0,1,\dots,m$□□□ K□□□ A□□□□□□□□□□□□□ K[A],□

$$K[A] \overset{\mathrm{def}}{=} \{a_m A^m + \dots + a_1 A + a_0 I \mid m \in N, a_i \in K, i=0,1,\dots,m\}$$

□ $f(A) = \sum_{i=0}^{m} a_i A^i$, $g(A) = \sum_{i=0}^{l} b_i A^i$. □□□ $m \geq l$. □□□□□□□□□□□

$$f(A) - g(A) = \sum_{i=0}^{m} (a_i - b_i) A^i, \tag{14}$$

$$f(A) g(A) = \sum_{s=0}^{m+l} \sum_{i+j=s} a_i b_j A^s. \tag{15}$$

□□ K[A]□ $M_n(K)$□□□□□.□□ $I \in K[A]$.□(15)□□□□□,$f(A)g(A) = g(A)f(A)$.□□ K[A]□□□□□□□□□.

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* □□□□□□□□□□□□□□□(□□)□(□□□□□□□ 1996□ 12□□ 1□)□ 8□.

$K[A]$□□□□□□□□□□□□ W, □□□□□□□□□□□□, □□ W□ $K[A]$□□□□□. □□ I□ W. □□□□□□□□ K□ W□□□□□□ □ $\tau: a \longmapsto aI$. □□ τ □□□, □□ τ □□□□□□□□□;

$$\tau(a+b)=(a+b)I=aI+bI=\tau(a)+\tau(b),$$

$$\tau(ab)=(ab)I=(aI)(bI)=\tau(a)\tau(b).$$

$K[A]$□□□□(14)□(15)□□□□□□ $K[x]$□□□□□□□□□ □. □□□□□□. □□□ $K[A]$□□□□□□□ $K[x]$□□□. □□□□□□ □□□□□□□□□□□□□□□ $K[x]$□□□□□:

□□ **3** □ K□□□□□, R□□□□□□□ 1□□□□□, R_1□ R□□ □□□, □ 1□□ R_1. K□ R_1 □□□□□ τ, □□ τ□□□□□□□□□□(□□□ $\tau(1)=1$□). □□□□ t□ R, □

$$\sigma_t: \quad K[x] \qquad\qquad R$$

$$f(x)=\sum_{i=0}^{n} a_i x^i \longmapsto \sum_{i=0}^{n} \tau(a_i)t^i \overset{\text{def}}{=} f(t),$$

□ σ_t □ $K[x]$□ R□□□; □□ σ_t □□□□□□□□□, □□□

$$f(x)+g(x)=h(x), \qquad f(x)g(x)=p(x), \tag{16}$$

□□□

$$f(t)+g(t)=h(t), \quad f(t)g(t)=p(t); \tag{17}$$

□□, $\sigma_t(x)=t$. □□□□□ σ_t □□ x□ t□□.

□□ □□ $K[x]$□□□□□ $f(x)$□□ $\sum_{i=0}^{n} a_i x^i$ □□□□□□(□□□□ □ 0□□□□), □□ τ □ K□ R_1 □□□, □□ σ_t □ $K[x]$□ R□□□□□.

□ σ_t □□□, □

$$\sigma_t(x)=\sigma_t(1x)=\tau(1)t=1t=t.$$

□ $f(x)=\sum_{i=0}^{n} a_i x^i$, $g(x)=\sum_{j=0}^{m} b_j x^j$, □□□ n□ m

□ $$h(x)=f(x)+g(x)=\sum_{i=0}^{n}(a_i+b_i)x^i,$$

$$p(x)=f(x)g(x)=\sum_{s=0}^{n+m}\Big(\sum_{i+j=s} a_i b_j\Big)x^s.$$

□ σ_t □□□, □

$$h(t)=\sum_{i=0}^{n}\tau(a_i+b_i)t^i=\sum_{i=0}^{n}[\tau(a_i)+\tau(b_i)]t^i$$

$$=\sum_{i=0}^{n}\tau(a_i)t^i+\sum_{i=0}^{n}\tau(b_i)t^i$$

$$= f(t) + g(t);$$

$$p(t) = \sum_{s=0}^{n+m} \tau\left(\sum_{i+j=s} a_i b_j\right) t^s = \sum_{s=0}^{n+m}\left(\sum_{i+j=s} \tau(a_i)\tau(b_j)\right) t^s;$$

$$f(t)g(t) = \left(\sum_{i=0}^{n} \tau(a_i) t^i\right)\left(\sum_{j=0}^{m} \tau(b_j) t^j\right)$$

$$= \sum_{i=0}^{n}\sum_{j=0}^{m} \tau(a_i)\tau(b_j) t^{i+j} = \sum_{s=0}^{n+m}\left(\sum_{i+j=s} \tau(a_i)\tau(b_j)\right) t^s$$

$$= p(t).$$

□□ σ_t □□□□□□□□□□. ■

□□ 3 □□□□, □□ R □□□□□□□□□, □ R □□□□□ R_1 □□□□ 3 □□□(□□□□□ R □□□□ K □□□□□), □□□□□□□□ $K[x]$ □□□□□□□□□□□□□□□, □□□□ x □ R □□□□□ t □□□□□□□. □□□□□□□□□□□□□ $K[x]$ □□□□□□□□□□□□□□□□, □□□□□ x □□ R □□□□□ t □□, □□□□□□ R □□□□□□□□□□. □□□□□□□□□ $K[x]$ □□□□□□□□.

□□□□□□□□, $K[x]$, $K[A]$(□□ A □ K □□□ n □□□)□□□□□□□ 3 □□□ R. □□, □□□ x □□□ x □□□□□□□□, □□□□□ A □□□□□□□□, □ $K[x]$ □□□□□□□□□□□□□□, □□ $K[x]$ □□□□□□□□□□□□□□, □□□□ $K[A]$ □□□□□□□□□□□.

□ 1 □ B □□□ K □ n □□□□□, □□□□□□ I. □ $A = I + kB$, k□ K. □□ A □□, □□□ A^{-1}.

□□ □ $K[x]$ □□□□□□

$$(1 - x)(1 + x + x^2 + \ldots + x^{l-1}) = 1 - x^l. \tag{18}$$

□□□ x □ $-kB$ □□, □(18)□□

$$(I + kB)[I - kB + (-kB)^2 + \ldots + (-kB)^{l-1}] = I - (-kB)^l.$$

□□ $B^l = 0$, □□□□□□

$$(I + kB)(I - kB + k^2 B^2 + \ldots + (-1)^{l-1} k^{l-1} B^{l-1}) = I. \tag{19}$$

(19)□□□, $I + kB$ □□, □□

$$(I + kB)^{-1} = I - kB + k^2 B^2 + \ldots + (-1)^{l-1} k^{l-1} B^{l-1}.$$ ■

□ □ 7.1

1. □ $K[x]$ □, □□ $f(x) = cg(x)$, □□ c□ K □ c□ 0, □□: $f(x)$ □ $g(x)$ □□□□□

□□□?

2. □ $K[x]$□,□□ $f(x)g(x)=c$,□□ c□ K□ c□ 0,□□:$f(x)$□ $g(x)$□□□□□□□?

3. □ $K[x]$□,□□ $f(x)$□ $g(x)$□□□□□ 3,□□:$f(x)+g(x)$□□□□□□ 3□?

4. □ R□□□□□□□ 1(□ 0)□□,□□ a□ R,□□□□ b□ R,□□

$$ab=ba=1,$$

□□ a□□□□(□□ a□□□,□□□□□□□□ 1□□),□ b□ a□□,□□ a^{-1}.

□□:$K[x]$□□□□□ $f(x)$□□□□□□□□ $f(x)$□□□□□□.

* 5. □ R□□□□□ 1(□ 0)□□,□□ R□□□□□□□□□□□□□□.

6. □

$$A=\begin{pmatrix} 1 & b & b^2 & \cdots & b^{n-2} & b^{n-1} \\ 0 & 1 & b & \cdots & b^{n-3} & b^{n-2} \\ \cdots & \cdots & \cdots & & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & b \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

□□ b□ K,□ A^{-1}.

7. □ A□ $M_n(K)$,□□□ A□□□□□□□

$$|\lambda I-A|=(\lambda-\lambda_1)^{l_1}(\lambda-\lambda_2)^{l_2}\cdots(\lambda-\lambda_s)^{l_s},$$

□□ $\lambda_1,\lambda_2,\cdots,\lambda_s$□□□□□□□□;$l_1+l_2+\cdots+l_s=n$.□□:□□ K□□□□□□ k,□□ kA□□□□□□□

$$|\lambda I-kA|=(\lambda-k\lambda_1)^{l_1}(\lambda-k\lambda_2)^{l_2}\cdots(\lambda-k\lambda_s)^{l_s}.$$

□□□□,□□ λ_i□ A□ l_i□□□□,□ $k\lambda_i$□ kA□ l_i□□□□.

* 8. □ A□ A□□□□□□□□ 7□□□□:A^2□□□□□□□

$$|\lambda I-A^2|=(\lambda-\lambda_1^2)^{l_1}(\lambda-\lambda_2^2)^{l_2}\cdots(\lambda-\lambda_s^2)^{l_s}.$$

□□□□,□□ λ_i□ A□ l_i□□□□,□ λ_i^2□ A^2□ l_i□□□□.

§2 □□□,□□□□

□□□□□□□□□□□□□□,□□□□□□□□□□□ $K[x]$□□□□□□□□□□□,□□□□□□□□□□□□ $K[x]$□□□.□□□□□□□□□□□□ $K[x]$□□□,□□ K□□□□□.

□□ $K[x]$□□□□□□ $f(x)$□ $g(x)$□□□□□□□:

$$f(x)=x^2-1,\quad g(x)=x-1.$$

□□,

$$f(x)=(x+1)g(x).$$

□□□□□□□□□□□:

□□ 1 □ $f(x),g(x)$□ $K[x]$,□□□□ $h(x)$□ $K[x]$,□□ $f(x)=$

$h(x)g(x)$,□□ $g(x)$□□ $f(x)$,□□ $g(x)\mid f(x)$;□□,□ $g(x)$□□□□ $f(x)$,□□ $g(x)\nmid f(x)$.

□ $g(x)$□□ $f(x)$□,$g(x)$□□ $f(x)$□□□,$f(x)$□□ $g(x)$□□□.

□□□□□□□□:

1° $0\mid f(x) \Longleftrightarrow f(x)=0$;

2° $f(x)\mid 0,\ \forall f(x)\in K[x]$;

3° $b\mid f(x),\ \forall b\in K$ □ $b\neq 0,\ \forall f(x)\in K[x]$.

□□□□□ K□□□□□□□□□□□□□ K^*.

$K[x]$□□□□□□□□□□□(□,$f(x)\mid f(x),\ \forall f(x)\in K[x]$),□□□(□,□□ $f(x)\mid g(x)$□ $g(x)\mid h(x)$,□ $f(x)\mid h(x)$),□□□□□□□□(□,□ $f(x)\mid g(x)$□□□ $g(x)\mid f(x)$).□□□□□□□□□□.

□□ **2** □ $K[x]$□,□□ $f(x)\mid g(x)$□ $g(x)\mid f(x)$,□□ $f(x)$□ $g(x)$□□,□□ $f(x)$□ $g(x)$.

□□ **1** □ $K[x]$□,$f(x)$□ $g(x)$□□□□□ $c\in K^*$,□□

$$f(x)=cg(x).$$

□□ □□□.□ $f(x)=cg(x)$,□□ $c\in K^*$.□ $g(x)\mid f(x)$.□□ $g(x)=\frac{1}{c}f(x)$,□□ $f(x)\mid g(x)$,□□ $f(x)$□ $g(x)$.

□□□.□ $f(x)$□ $g(x)$.□ $f(x)\mid g(x)$□ $g(x)\mid f(x)$.□□□□ $h_1(x)$,$h_2(x)\in K[x]$,□□

$$g(x)=h_1(x)f(x),\qquad f(x)=h_2(x)g(x).$$

□□

$$f(x)=h_2(x)h_1(x)f(x). \tag{1}$$

□□ $f(x)=0$,□ $g(x)=0$.□□ $f(x)$□ $g(x)$.□□□ $f(x)\neq 0$.□□□□□,□(1)□□

$$1=h_2(x)h_1(x). \tag{2}$$

□(2)□□,

$$\deg h_2(x)+\deg h_1(x)=0.$$

□□,$\deg h_1(x)=\deg h_2(x)=0$,□□ $h_2(x)=c$,$c\in K^*$.□□ $f(x)=cg(x)$.

□□ **2** □ $K[x]$□,□□ $g(x)\mid f_i(x)$,$i=1,2,\ldots,s$,□□□□□ $u_i(x)\in K[x]$,$i=1,2,\ldots,s$,□

$$g(x)\mid (u_1(x)f_1(x)+u_2(x)f_2(x)+\cdots+u_s(x)f_s(x)).$$

□□ 2□□□□□□□.

$K[x]$□,□□ $g(x)$□□□□ $f(x)$,□□□□□□□□□□□?□□,□ $f(x)=x^2$,$g(x)=x-1$.□

$$f(x)=x^2-1+1=(x+1)g(x)+1.$$

□□□□□□,□□□□□□□:

□□ **3(□□□□)** □□ $K[x]$□□□□□□□□ $f(x)$□ $g(x)$,□□ $g(x)$□0,□ $K[x]$□□□□□□□□□□□ $h(x)$, $r(x)$,□□

$$f(x)=h(x)g(x)+r(x),\quad \deg r(x)<\deg g(x). \tag{3}$$

□□ □□□.□ $\deg g(x)=m$□N.

□□1. $m=0$.□□ $g(x)=b$, b□K^*.□□

$$f(x)=\frac{1}{b}f(x)\;b+0,\quad \deg 0<\deg g(x),$$

□□2. $m>0$,□ $\deg f(x)<m$.□

$$f(x)=0\,g(x)+f(x),\quad \deg f(x)<\deg g(x).$$

□□3. $m>0$,□ $\deg f(x)$□ m□□□□□□□ n□□□□□□.□□□□□□□□ n□□□□,□□□□□□□□□□.□□□□ n□□□□ $f(x)$.□ $f(x)$, $g(x)$□□□□□□ a_nx^n, b_mx^m.□□ $a_nb_m^{-1}x^{n-m}g(x)$□□□□ a_nx^n.□

$$f_1(x)=f(x)-a_nb_m^{-1}x^{n-m}g(x), \tag{4}$$

□ $\deg f_1(x)<n$.□□□□□□,□□ $h_1(x)$, $r_1(x)$□ $K[x]$,□□

$$f_1(x)=h_1(x)g(x)+r_1(x),\quad \deg r_1(x)<\deg g(x). \tag{5}$$

□(5)□□□(4)□,□

$$\begin{aligned} f(x)&=f_1(x)+a_nb_m^{-1}x^{n-m}g(x)\\ &=[h_1(x)+a_nb_m^{-1}x^{n-m}]g(x)+r_1(x). \end{aligned} \tag{6}$$

□ $h(x)=h_1(x)+a_nb_m^{-1}x^{n-m}$,□

$$f(x)=h(x)g(x)+r_1(x),\quad \deg r_1(x)<\deg g(x). \tag{7}$$

□□□□□□□□□,□□3□□□□□□□□.

□□□.□ $h(x)$, $r(x)$, $h\square(x)$, $r\square(x)$□ $K[x]$,□□

$$f(x)=h(x)g(x)+r(x),\quad \deg r(x)<\deg g(x), \tag{8}$$

$$f(x)=h\square(x)g(x)+r\square(x),\quad \deg r\square(x)<\deg g(x) \tag{9}$$

□□(8),(9)□□□

$$h(x)g(x)+r(x)=h\square(x)g(x)+r\square(x),$$

□

$$[h(x)-h\square(x)]g(x)=r\square(x)-r(x). \tag{10}$$

□□

$$\begin{aligned} &\deg[h(x)-h\square(x)]+\deg g(x)=\deg[r\square(x)-r(x)]\\ &\square \max\{\deg r\square(x),\deg r(x)\}<\deg g(x). \end{aligned} \tag{11}$$

□□ $h(x)$□$h\square(x)$,□□(11)□□

$$\deg[h(x)-h\square(x)]<0,$$

□□.□□ $h(x)=h\square(x)$.□□ $r(x)=r\square(x)$.□□□□□. x

(3) □□□ $h(x)$□□ $g(x)$□ $f(x)$□□□, $r(x)$□□ $g(x)$□ $f(x)$□□□.

□□4 □ $f(x), g(x)$□ $K[x]$, □ $g(x)$□0, □ $g(x)\mid f(x)$□□□□ $g(x)$□ $f(x)$□□□□□.

□□ $g(x)\mid f(x)$ � $f(x)=h(x)g(x)$, $h(x)$□ $K[x]$.

� $g(x)$□ $f(x)$□□□□ 0. x

□□□□□□□□□□: □□ $K[x]$□□□□□ $f(x), g(x)$, □□□ $K[x]$□, $g(x)$□□□□ $f(x)$, □□□□□ K□□□□□ F□, □ $F[x]$□, $g(x)$□□□□□□ $f(x)$. □□□□□□□□:

□□5 □ $f(x), g(x)$□ $K[x]$, □□ F� K, □

□ $K[x]$□, $g(x)\mid f(x)$ � □ $F[x]$□, $g(x)\mid f(x)$.

□□ □□□. □□ $K[x]$□, $g(x)\mid f(x)$, □□□ $h(x)$□ $K[x]$, □□ $f(x)=h(x)g(x)$. □□ K� F, □□ $f(x), g(x), h(x)$□ $F[x]$. □□□ $F[x]$□, $g(x)\mid f(x)$.

□□□. □□ $F[x]$□, $g(x)\mid f(x)$.

□ $g(x)$□0□, □ $K[x]$□□□□□□, □ $h(x), r(x)$□ $K[x]$, □□

$$f(x)=h(x)g(x)+r(x),\quad \deg r(x)<\deg g(x). \qquad (12)$$

□□ $f(x), g(x), h(x), r(x)$□ $F[x]$, □□(12)□□□□□□□ $F[x]$□□□□□□. □□□ $F[x]$□, $g(x)\mid f(x)$, □□□□□□ 4□, $r(x)=0$. □□□ $K[x]$□, $g(x)\mid f(x)$.

□ $g(x)=0$□, □ $g(x)\mid f(x)$□, $f(x)=0$. □□□ $K[x]$□, □ $g(x)\mid f(x)$. x

□□□□□□,□□□□□□□ 1□□□□□.

□1 □ $g(x)$□ $f(x)$, □□□□□□, □□

$$f(x)=2x^3+3x^2+5,\quad g(x)=x^2+2x-1.$$

□

$$\begin{array}{r|l|l}
x^2+2x-1 & 2x^3+3x^2+5 & 2x-1 \\
 & 2x^3+4x^2-2x & \\
\hline
 & -x^2+2x+5 & \\
 & -x^2-2x+1 & \\
\hline
 & 4x+4 &
\end{array}$$

□□

$$2x^3+3x^2+5=(2x-1)(x^2+2x-1)+(4x+4),$$

□□ $g(x)$□□ $f(x)$, □□□ $2x-1$, □□□ $4x+4$.

□□□□□,□□□□□□□□□,□□□□□□□□□□□[1].

□ **2** □ $f(x)=2x^4-6x^3+3x^2-2x+5$, $g(x)=x-2$, □ $g(x)$□ $f(x)$ □□□□□□□□.

□

$$\begin{array}{ccccc|c}
2 & -6 & 3 & -2 & 5 & 2 \\
 & 4 & -4 & -2 & -8 & \\
\hline
2 & -2 & -1 & -4 & -3 &
\end{array}$$

□□ $f(x)=(2x^3-2x^2-x-4)(x-2)-3$. □□ $x-2$ □□ $f(x)$, □□□ $2x^3-2x^2-x-4$, □□□ -3.

□□ 7.2

1. □□□□□□□□□□, □□ $K[x]$□, □□ $f(x)|g(x)$□ $g(x)|h(x)$, □ $f(x)|h(x)$.

2. □□□□□□□ 2, □□ $K[x]$□, □□

$$g(x)|f_i(x),\quad i=1,2,\ldots,s,$$

□□□□□ $u(x)$□ $K[x]$, $i=1,2,\ldots,s$, □

$$g(x)|(u_1(x)f_1(x)+u_2(x)f_2(x)+\cdots+u_s(x)f_s(x)).$$

3. □ $g(x)$□ $f(x)$, □□□□□□.

(1) $f(x)=x^4-3x^2-2x-1$, $g(x)=x^2-2x+5$;

(2) $f(x)=x^4+x^3-2x+3$, $g(x)=3x^2-x+2$.

4. □ $f(x)=x^4-3x^3+a_1x+a_0$, $g(x)=x^2-3x+1$, □ $g(x)$□□ $f(x)$□□□□□□□.

5. □□□□□□□□□□□ $g(x)$□ $f(x)$□□□□□□□□.

(1) $f(x)=3x^4-5x^2+2x-1$, $g(x)=x-4$;

(2) $f(x)=5x^3-3x+4$, $g(x)=x+2$.

*6. □ a, b□ $\mathbf{Z}$, □□□ h□ $\mathbf{Z}$□□

$$a=hb,$$

[1] □□□□□□□□□□□,□□□□□□□□□□□□(□□)□(□□□□□□□ 1996□ 12□ □ 1□)□ 23□.

□□ b□□ a, □□ $b|a$, □□ b□□ a□□□(□□□), a□□ b□□□; □□, □ b□□□□ a, □□ $b∤a$. □□:

(1) □□ $a|b$□ $b|a$(□□□ a□ b□□), □ $a=\pm b$; □□□□□;

(2) □□ $a|b$□ $b|c$, □ $a|c$;

(3) □□ $b|a_i$, $i=1,2,\ldots,s$, □□□□□ u_i□ Z, $i=1,2,\ldots,s$, □

$$b|(u_1a_1+u_2a_2+\ldots+u_sa_s).$$

(4) □□ $b|a$, □ a□ 0, □ $|b|$□$|a|$.

□□□□□:□□□□□□□□□

□□ 1 □□ a, b□ Z, b□ 0, □□□□□□□□□ q, r, □□

$$a=bq+r,\ 0\square r<|b|.$$

□□ □□□□

$$S=\{a-bs\,|\,s\square Z,\ a-bs\square 0\}.$$

□ $b>0$, □ $a-b(-a^2)=a+a^2b\square 0$; □ $b<0$, □ $a-ba^2\square 0$. □□ S□□, □□ S□□□□□□□□□□, □□ $a-bq$. □ $r=a-bq$, □□ S□□□, $r\square 0$. □□ $r\square|b|$, □ $b>0$, □ $r-b\square 0$, □□ $a-bq-b\square 0$, □ $a-b(q+1)\square 0$. □□ $a-b(q+1)\square S$, □ $r-b\square S$, □ $r-b<r$, □□ r□□□□□. □□ r□ $|b|$, □ $b<0$, □ $0\square r+b=a-b(q-1)$. □□ $r+b\square S$, □ $r+b<r$, □□. □□ $r<|b|$. □□□□□.

□□□. □□□□□□ $q\square, r\square$, □□

$$a=bq\square+r\square,\ 0\square r\square<|b|,$$

□□ $bq+r=bq\square+r\square$. □□□ $r\square r\square$. □□ $b(q-q\square)=r\square-r$. □□ $0\square r<|b|$, $0\square r\square<|b|$, $r\square r\square$, □□ $0\square r\square-r<|b|$. □□

$$|q-q\square|=\frac{r\square-r}{|b|}<1,$$

□□□ $q-q\square=0$, □ $q=q\square$. □□□ $r=r\square$. x

§ 3 □□□□□

□□□□□□, □□ K□□□□□□□ $K[x]$□□□□□, □□ $K[x]$□□□□□□. □□□□□□□□□□□□□ $K[x]$□□□□□□□: $K[x]$□□□□□□□□□□□□□□□, □□ $f(x)$□ $g(x)$□□□□□□□□□ $f(x)$□ $g(x)$□□□□□.

□ $K[x]$□,□□ $c(x)$□□ $f(x)$□□□,□□ $g(x)$□□□,□□ $c(x)$□ $f(x)$□ $g(x)$□□□□□□. □ $f(x)$□ $g(x)$□□□□□□,□□□□□□□□□□□□□□,□

□□ 1 $K[x]$□□□□ $f(x)$□ $g(x)$□□□□□ $d(x)$□□□□□□□□;□□ $f(x)$□ $g(x)$□□□□□ $c(x)$,□□ $c(x)\mid d(x)$,□□ $d(x)$□ $f(x)$□ $g(x)$□□□□□□□.

□□□□□□□,□□□□□□□□□□□□□□ 0;□□□□□□□ $f(x)$,$f(x)$□ $f(x)$□ 0□□□□□□□□.

□□ $K[x]$□□□□□□□□,□□□□□□□□□□□□?□□□□,□□□□□□□□□□□□□?□□□□□□□□□□ $f(x)$□ $g(x)$,□□□□□□□□□□□□?□□□□□□□□□□□□.

□□□□□□□□□□□□□□□:

□□ 1 □ $f(x),g(x),p(x),q(x)$□□ $K[x]$□□□□□,□□ $f(x)$□ $g(x)$□□□□□□□□□□□□□ $p(x)$□ $q(x)$□□□□□□□□□□□,□□ $f(x)$□ $g(x)$□□□□□□□□□□□ $p(x)$□ $q(x)$□□□□□□□□.

□□ □ $d(x)$□ $f(x)$□ $g(x)$□□□□□□□□,□ $d(x)$□ $p(x)$□ $q(x)$□□□□□□. □□ $p(x)$□ $q(x)$□□□□□□ $\varphi(x)$,□ $\varphi(x)$□□ $f(x)$□ $g(x)$□□□□,□□ $\varphi(x)\mid d(x)$. □□ $d(x)$□ $p(x)$□ $q(x)$□□□□□□□□. □□, $p(x)$□ $q(x)$□□□□□□□□□□ $f(x)$□ $g(x)$□□□□□□. x

□□ 2 □ $f(x),g(x)$□ $K[x]$,a,b□ K□□□□,□ $f(x)$□ $g(x)$□□□□□□□□□□□ $af(x)$□ $bg(x)$□□□□□□□□□.

□□ □□ $f(x)$□ $g(x)$□□□□□□□ $af(x)$□ $bg(x)$□□□□. □□□□ $af(x)$□ $bg(x)$□□□□□□□□ $f(x)$□ $g(x)$□□□□. □□□□□ 1 □□□□□□. x

□□ 1 □ $K[x]$□,□□□□□

$$f(x)=h(x)g(x)+r(x) \tag{1}$$

□□,□ $f(x)$□ $g(x)$□□□□□□□□□□□ $g(x)$□ $r(x)$□□□□□□□□□.

□□ □ $d(x)$□ $f(x)$□ $g(x)$□□□□□□,□ $d(x)\mid f(x)$,□□ $d(x)\mid g(x)$. □□□(1)□□

$$r(x)=f(x)-h(x)g(x),$$

□□ $d(x)\mid r(x)$. □ $d(x)$□ $g(x)$□ $r(x)$□□□□□□. □□□□ $g(x)$□ $r(x)$□□□□□□ $c(x)$,□(1)□□,$c(x)\mid f(x)$. □□ $c(x)$□ $f(x)$□ $g(x)$

□□□□. □□□ 1 □□□□□□□□□□. x

□□□□□□□□□□□□□:

□□ 3 □□ $K[x]$ □□□□□□□ $f(x)$ □ $g(x)$, □□□□□□□□
□□□□ $d(x)$, □□ $d(x)$ □□□□□ $f(x)$ □ $g(x)$ □□□□□, □□ $K[x]$
□□□□ $u(x)$ □ $v(x)$, □

$$d(x)=u(x)f(x)+v(x)g(x). \tag{2}$$

□□ □□ $g(x)=0$, □ $f(x)$ □□ $f(x)$ □ $g(x)$ □□□□□□□□, □□

$$f(x)=1\cdot f(x)+1\cdot 0.$$

□□□ $g(x)$ □ 0. □□□□□, □ $h_1(x), r_1(x)$ □ $K[x]$, □

$$f(x)=h_1(x)g(x)+r_1(x), \quad \deg r_1(x)<\deg g(x).$$

□□ $r_1(x)$ □ 0, □□ $r_1(x)$ □□ $g(x)$, □ $h_2(x), r_2(x)$ □ $K[x]$, □□

$$g(x)=h_2(x)r_1(x)+r_2(x), \quad \deg r_2(x)<\deg r_1(x).$$

□□□ $r_2(x)$ □ 0, □□ $r_2(x)$ □□ $r_1(x)$, □ $h_3(x), r_3(x)$ □ $K[x]$, □□

$$r_1(x)=h_3(x)r_2(x)+r_3(x), \quad \deg r_3(x)<\deg r_2(x).$$

□□□□□□□□, □□, □□□□□□□□□□, □□□□□□□□, □□□
□□□□, □

$$r_2(x)=h_4(x)r_3(x)+r_4(x), \quad \deg r_4(x)<\deg r_3(x),$$

....

$$r_{i-2}(x)=h_i(x)r_{i-1}(x)+r_i(x), \quad \deg r_i(x)<\deg r_{i-1}(x),$$

....

$$r_{s-3}(x)=h_{s-1}(x)r_{s-2}(x)+r_{s-1}(x), \quad \deg r_{s-1}(x)<\deg r_{s-2}(x),$$

$$r_{s-2}(x)=h_s(x)r_{s-1}(x)+r_s(x), \quad \deg r_s(x)<\deg r_{s-1}(x),$$

$$r_{s-1}(x)=h_{s+1}(x)r_s(x)+0,$$

□□□□ $h_i(x), r_i(x)$ □ $K[x]$. □□□□□□□□□□□□□□, $r_s(x)$ □
$r_{s-1}(x)$ □ $r_s(x)$ □□□□□□□□. □□□ 1 □, $r_s(x)$ □□ $r_{s-2}(x)$ □
$r_{s-1}(x)$ □□□□□□□□; □□ $r_s(x)$ □□ $r_{s-3}(x)$ □ $r_{s-2}(x)$ □□□□□□
□□; □□□□□, $r_s(x)$ □□ $f(x)$ □ $g(x)$ □□□□□□□□. □□□□: □□
$f(x)$ □ $g(x)$ □□□□□, □□□□□□□□□□□□ $f(x)$ □ $g(x)$ □□□
□□□□□.

□□□□□□□□□□□□□

$$r_s(x)=r_{s-2}(x)-h_s(x)r_{s-1}(x),$$

□□□□□□□□

$$r_{s-1}(x)=r_{s-3}(x)-h_{s-1}(x)r_{s-2}(x),$$

□□□□□

$$r_s(x)=(1+h_s(x)h_{s-1}(x))r_{s-2}(x)-h_s(x)r_{s-3}(x).$$

□□□□□□□□□□□□□□ $r_{s-2}(x), r_{s-3}(x), \ldots, r_1(x)$,□□

$$r_s(x)=u(x)f(x)+v(x)g(x),$$

□□ $u(x), v(x)$□ $K[x]$. x

□□ 3□□□□□□□□□□□□□□□□□□□□□,□□□□□□□
□.

□□□□ $K[x]$□□□□□□ $f(x)$□ $g(x)$,□□□□□□□□□□□□ ?
□ $d_1(x), d_2(x)$□□ $f(x)$□ $g(x)$□□□□□□,□□□□, $d_1(x)|d_2(x)$□
$d_2(x)|d_1(x)$.□□ $d_1(x)$□ $d_2(x)$□□,□ $d_1(x)$□ $d_2(x)$□□□□□□□
□□□.□□□:□□□□□□□□□□□□□□□□□□□□□□□□□.□□
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□□□,□

$$(f(x), g(x))$$

□□□□□□□□ 1□□□□□□□□.

□□□ 2□□,□ $f(x), g(x)$□□□□,□□□ a, b□ K□ a□0, b□0,□

$$(f(x), g(x))=(af(x), bg(x)).$$

□ 1 □

$$f(x)=x^3+x^2-7x+2, \quad g(x)=3x^2-5x-2,$$

□$(f(x), g(x))$,□□□□□□□ $f(x)$□ $g(x)$□□□□□.

□ □□□□□□,□□□□□□□□,□□□□□□□□□□□□□□□
□□□,□□□□□□□□□.

$h_2(x)=3x+1$	$g(x)$	$3f(x)$	
	$3x^2-5x-2$	$3x^3+3x^2-21x+6$	$x+\frac{8}{3}=h_1(x)$
	$3x^2-6x$	$3x^3-5x^2-2x$	
	$x-2$	$8x^2-19x+6$	
	$x-2$	$8x^2-\frac{40}{3}x-\frac{16}{3}$	
	0	$r_1(x)=-\frac{17}{3}x+\frac{34}{3}$	
		$-\frac{3}{17}r_1(x)=x-2$	

□□□□□□□□□□□□□□ $r_1(x)$,□□

$$(f(x), g(x))=x-2.$$

□□□□□□□□□□□□□□

$$3f(x) = \left(x + \frac{8}{3}\right) g(x) + r_1(x),$$

$$g(x) = (3x+1)\left(-\frac{3}{17} r_1(x)\right) + 0.$$

□□
$$(f(x), g(x)) = -\frac{3}{17} r_1(x)$$
$$= -\frac{3}{17}\left[3f(x) - \left(x + \frac{8}{3}\right) g(x)\right]$$
$$= -\frac{9}{17} f(x) + \frac{1}{17}(3x+8) g(x).$$

□□□□□□□□□□□□□□□□□□□□□□(□□□□□□),□□□□□□□,□□□□□□□□.

□□ 2 $K[x]$□□□□□□□ $f(x)$, $g(x)$, □□ $(f(x), g(x)) = 1$, □□ $f(x)$□ $g(x)$□□.

□□,□□□□□□□□,□□□□□□□□□□□□□□□□□□□,□□□□.

□□□□□□□□□□□□□□□□□□□□□,□□□□□.

□□ 4 $K[x]$□□□□□□ $f(x)$□ $g(x)$□□□□□□□□□□□ $K[x]$□□□□□ $u(x)$, $v(x)$, □

$$u(x) f(x) + v(x) g(x) = 1. \tag{3}$$

□□ □□□.□□□ 3□□□□.

□□□□□□.□(3)□□□.□□ $(f(x), g(x)) \mid f(x)$, □□ $(f(x), g(x)) \mid g(x)$, □□□(3)□□, $(f(x), g(x)) \mid 1$.□□

$$(f(x), g(x)) = 1.$$ x

□□□□ 4□□□□□□□□□□□□□□□□□□□□:

□□ 1 □ $K[x]$□,□□

$$f(x) \mid g(x) h(x),\ □\ (f(x), g(x)) = 1,$$

□
$$f(x) \mid h(x).$$

□□ □ $h(x) = 0$, □ $f(x) \mid h(x)$.□□□ $h(x)$□ 0.□□ $(f(x), g(x)) = 1$, □□□ $u(x)$, $v(x)$□ $K[x]$,□□

$$u(x) f(x) + v(x) g(x) = 1$$

□□□□□□ $h(x)$,□

$$u(x) f(x) h(x) + v(x) g(x) h(x) = h(x), \tag{4}$$

□□ $f(x) \mid g(x) h(x)$,□□ $f(x)$□□(4)□□□,□□

$$f(x) \mid h(x).$$ x

□□ 2 □ $K[x]$□,□□

$$f(x)\mid h(x),\quad g(x)\mid h(x),\quad □\ (f(x),\ g(x))=1,$$

□

$$f(x)g(x)\mid h(x).$$

□□ □□ $f(x)\mid h(x)$, □□□ $p(x)$□ $K[x]$, □

$$h(x)=p(x)f(x). \tag{5}$$

□□ $g(x)\mid h(x)$, □□ $g(x)\mid p(x)f(x)$. □□ $(g(x),\ f(x))=1$, □□ $g(x)\mid p(x)$. □□□ $q(x)$□ $K[x]$, □□ $p(x)=q(x)g(x)$. □□(5)□□

$$h(x)=q(x)g(x)f(x),$$

□□ $f(x)g(x)\mid h(x)$. x

□□ 3 □ $K[x]$□, □□

$$(f(x),\ h(x))=1,\quad (g(x),\ h(x))=1,$$

□

$$(f(x)g(x),\ h(x))=1$$

□□ □□ $(f(x),\ h(x))=1$, $(g(x),\ h(x))=1$, □□□ $K[x]$□□□□ □ $u_i(x)$, $v_i(x)$, $i=1,2$, □□

$$u_1(x)f(x)+v_1(x)h(x)=1,$$

$$u_2(x)g(x)+v_2(x)h(x)=1.$$

□□□□□□□□□, □

$$u_1(x)u_2(x)f(x)g(x)+[u_1(x)f(x)v_2(x)$$
$$+v_1(x)u_2(x)g(x)+v_1(x)v_2(x)h(x)]h(x)=1,$$

□□□ 4□, $(f(x)g(x),\ h(x))=1$. x

□□□□□□□□□□□□□□□□ $n(n>2)$ □□□□□□.

□□ 3 □ $K[x]$□, □□□□□ $c(x)$□□□□□ $f_1(x), f_2(x), \ldots, f_n(x)$□□□□, □□ $c(x)$□□□ n □□□□□□□□□□. □ $d(x)$□ $f_1(x), \ldots, f_n(x)$□□□□□□, □□ $d(x)$□□□□□□: $f_1(x), \ldots, f_n(x)$□□□□□□□□□□ $d(x)$, □ $d(x)$□□ $f_1(x), \ldots, f_n(x)$□□□□□□□.

□□□□□□□□□□, □ $K[x]$□, □□ n□(n□2)□□□ $f_1(x), \ldots, f_n(x)$□□□□□□□□□, □□□□ $d_1(x)$□ $f_1(x), \ldots, f_{n-1}(x)$□□□□□□□□, □ $d_1(x)$□ $f_n(x)$□□□□□□□ $f_1(x), \ldots, f_{n-1}(x), f_n(x)$□□□□□□. □□□□□□□□□□□□□□ n□□□□□□□□□□□□.

□□□ 3□□□□, n□□□□ $f_1(x), \ldots, f_n(x)$□□□□□□□□□□□□□□□□□. □□ n□□□□□□□□□ $f_1(x), \ldots, f_n(x)$, □□□□, □

$$(f_1(x),\ f_2(x),\ \ldots,\ f_n(x))$$

□□□□□□□ 1□□□□□□□□. □□□□□□

$$(f_1(x), f_2(x), \ldots, f_n(x))=((f_1(x), \ldots, f_{n-1}(x)), f_n(x)). \tag{6}$$

(6) 3 , $K[x]$ $u_i(x)$, $i=1,\dots,n$,

$$u_1(x)f_1(x)+u_2(x)f_2(x)+\dots+u_n(x)f_n(x)$$
$$=(f_1(x),f_2(x),\dots,f_n(x)). \tag{7}$$

4 $K[x]$ $n(n\geq 2)$ $f_1(x),f_2(x),\dots,f_n(x)$, $(f_1(x),f_2(x),\dots,f_n(x))=1$, $f_1(x),f_2(x),\dots,f_n(x)$.

4 : $K[x]$, n $f_1(x),f_2(x),\dots,f_n(x)$ $K[x]$ $u_1(x),u_2(x),\dots,u_n(x)$,

$$u_1(x)f_1(x)+u_2(x)f_2(x)+\dots+u_n(x)f_n(x)=1. \tag{8}$$

: $n(n>2)$, . ,

$$f_1(x)=x+1,\ f_2(x)=x^2+3x+2,\ f_3(x)=x-1.$$

$$(f_1(x),f_2(x))=x+1,\ (f_1(x),f_2(x),f_3(x))=1,$$

, $f_1(x),f_2(x),f_3(x)$, $f_1(x)$ $f_2(x)$.

, K F , K F. $f(x)$, $g(x)$ $K[x]$, $f(x)$ $g(x)$ $F[x]$. $f(x)$ $g(x)$ $K[x]$ $F[x]$,

$$f(x)=x^2+1,\quad g(x)=x^3+x^2+x+1,$$

$f(x)$ $g(x)$ $R[x]$, $f(x)$ $g(x)$ $C[x]$ $x+i$ $x-i$. $R[x]$ x^2+1, $C[x]$ x^2+1. , :

5 F K, $K[x]$ $f(x)$ $g(x)$, $K[x]$ 1 $F[x]$ 1 , , , $f(x)$ $g(x)$ 1 .

$f(x)=g(x)=0$, $f(x)$ $g(x)$ $K[x]$, $F[x]$.

$f(x)$ $g(x)$. $d_1(x)$ $f(x)$ $g(x)$ $K[x]$ 1 , $d_2(x)$ $f(x)$ $g(x)$ $F[x]$ 1 . $K[x]$ $f(x)$ $g(x)$, $r_s(x)$, c, $d_1(x)=\frac{1}{c}r_s(x)$, $F[x]$ ( ), $r_s(x)$ $f(x)$ $g(x)$ $F[x]$,

$$d_2(x)=\frac{1}{c}r_s(x)=d_1(x).$$

x

□□6 □□□ F� K. □ $f(x), g(x)$□ $K[x]$, □ $f(x)$□ $g(x)$□ $K[x]$□□□,□□□□ $f(x)$□ $g(x)$□ $F[x]$□□□.□□□□□□□□□□□□□□.

□□ $f(x)$□ $g(x)$□ $K[x]$□□□

� □ $K[x]$□, $(f(x), g(x)) = 1$

� □ $F[x]$□, $(f(x), g(x)) = 1$

� $f(x)$□ $g(x)$□ $F[x]$□□□. x

□ □ 7.3

1. □ $f(x)$□ $g(x)$□□□□□□,□□□ $(f(x), g(x))$□□□ $f(x)$□ $g(x)$□□□□ □:

(1) $f(x) = x^4 + 3x^3 - x^2 - 4x - 3$,

$g(x) = 3x^3 + 10x^2 + 2x - 3$;

(2) $f(x) = x^4 + 6x^3 - 6x^2 + 6x - 7$,

$g(x) = x^3 + x^2 - 7x + 5$.

2. □□:□ $K[x]$□,□□ $d(x)$□ $f(x)$□ $g(x)$□□□□□,□□ $d(x)$□ $f(x)$□ $g(x)$□□□□□□,□ $d(x)$□ $f(x)$□ $g(x)$□□□□□□□□.

3. □□:□ $K[x]$□, $(f, g)h$□ fh□ gh□□□□□□□□;□□□,□ $h(x)$□□□□□□ 1,□

$$(fh, gh) = (f, g)h.$$

4. □□:□ $K[x]$□,□□ $f(x), g(x)$□□□□,□

$$\left(\frac{f(x)}{(f(x), g(x))}, \frac{g(x)}{(f(x), g(x))}\right) = 1.$$

5. □□:□ $K[x]$□,□□ $f(x), g(x)$□□□□,□□

$$u(x)f(x) + v(x)g(x) = (f(x), g(x)),$$

□ $(u(x), v(x)) = 1$.

6. □□:□ $K[x]$□,□□ $(f, g) = 1$,□□

$$(fg, f + g) = 1.$$

7. □ $f(x), g(x)$□ $K[x]$,□□ a, b, c, d□ K,□□

$$ad - bc \neq 0.$$

□□: $(af + bg, cf + dg) = (f, g)$.

8. □□:□ $K[x]$□,□□ $(f(x), g(x)) = 1$,□□□□□□□ m□

$$(f(x^m), g(x^m)) = 1.$$

9. □□: $K[x]$□□□□□□□□ $f(x)$□ $g(x)$□□□□□□□□□□□,□□□□□ □□□□ $u(x), v(x)$,□

$$u(x)f(x)=v(x)g(x),$$

$$\deg u(x)<\deg g(x),\quad \deg v(x)<\deg f(x).$$

10. □ $f(x), g(x)$□ $K[x]$, $K[x]$□□□□□□ $m(x)$□□ $f(x)$□ $g(x)$□□□□□□□□□,□□

1) $f(x)\mid m(x)$, $g(x)\mid m(x)$;

2) $f(x)$□ $g(x)$□□□□□□(□ $K[x]$□□□□ $f(x)$□□,□□□ $g(x)$□□□□□□)□□ $m(x)$□□□.

(1) □□ $K[x]$□□□□□□□□□□□□□,□□□□□□□□□□□□□;

(2) □ $[f(x), g(x)]$□□□□□□ 1□□□□□□□.□□:□□ $f(x), g(x)$□□□□□□□ 1,□

$$[f(x), g(x)]=\frac{f(x)g(x)}{(f(x), g(x))}.$$

*11. □ A□ $M_n(K)$, $f(x), g(x)$□ $K[x]$.□□:□□ $d(x)$□ $f(x)$□ $g(x)$□□□□□□□□,□□□□□□□ $d(A)X=0$□□□□□ $f(A)X=0$□□□□ $g(A)X=0$□□□□□.

*12. □ A□ $M_n(K)$, $f_1(x), f_2(x)$□ $K[x]$,□ $f(x)=f_1(x)f_2(x)$.□□:□□ $(f_1(x), f_2(x))=1$,□ $f(A)X=0$□□□□□□□□□□□□ $f_1(A)X=0$□□□□□ $f_2(A)X=0$□□□□□.

□□□□□:□□□□□

□□□□ Z□□□□□□,□□□□□□□□□□□□□□□□□.□□□□□.

□□ **1** □ a, b□□□,□□□ c□□ a□□□,□□ b□□□,□□ c□ a□ b□□□□□(□□□□).□ d□ a□ b□□□□□,□□ a□ b□□□□□□□□□ d,□□ d□ a□ b□□□□□□□(□□□□□□).

□□ **1** □ Z□,□□□□□

$$a=bq+c$$

□□,□ a□ b□□□□□□ b□ c□□□□□□,□□□□.

□□ □□□§3□□□ 1□□□. X

□□ **1** □□□□□□□□ a, b,□□□□□□□□□□□□□ d,□□ d□□□□□ a□ b□□□□□,□□□□□ u, v,□□

$$d=ua+vb. \tag{1}$$

□□ □□ $b=0$,□ a□□ a□ b□□□□□□□,□□

$$a=1\cdot a+1\cdot 0.$$

□□□ b□ 0.□□□□ a□ b□□□□□□□ a□ $-b$□□□□□□,□

□□□.□□□□□□□ $b>0$.□□□□□,□□□ q_1, r_1,□□

$$a = bq_1 + r_1, 0□ r_1 < b.$$

□□ r_1□0,□□ r_1□□ b,□□□ q_2, r_2,□□

$$b = r_1 q_2 + r_2, 0□ r_2 < r_1.$$

□□ r_2□0,□□ r_2□□ r_1.□□□□□□□,□□□□□(□□□)□□□□,□□□□□□,□□□□□□.□□□□□□ 3□□□□□□□,□□□□□□□□□□□□□□□ a□ b□□□□□□□□,□□□□□□□□ a□ b□□□□□. X

□□□ 1□□,a□ b□□□□□□□□□□□□□(□□□□□□□□)□□□□.□□□□□□□□□ a, b,□□□□□(a, b)□□□□□□□□□□□.

□□ **2** □ a, b□Z,□□$(a, b) = 1$,□□ a□ b□□.

□□,□□□□□□□□,□□□□□□ ± 1□□□□□□□□□,□□□□.

□□ **2** □□□□ a□ b□□□□□□□□□□□□□ u, v,□□

$$ua + vb = 1.$$

□□ □□□□□□ 1□□□□.□□□□□. X

□□□ 2□□□□

□□ **3** □ Z□,□□ $a | bc$,□□$(a, b) = 1$,□ $a | c$.

□□□□□□.

□□ **3** □ Z□,□□ c□ $a_1, a_2, \ldots, a_n$ □□□□□□□,□□ c□ $a_1, a_2, \ldots, a_n$□□□□□□.□ d□ $a_1, a_2, \ldots, a_n$ □□□□□,□□ $a_1, a_2, \ldots, a_n$ □□□□□□□□□□□ d,□□ d□ $a_1, a_2, \ldots, a_n$ □□□□□□□□.

□□□□□□□□□□,□□ $n(n□2)$□□□□□□□□□□,□□□□ d_1 □ $a_1, a_2, \ldots, a_{n-1}$ □□□□□□□□,□ d_1 □ a_n □□□□□□□ $a_1, a_2, \ldots, a_{n-1}, a_n$ □□□□□□.□□□□□□□□□□□□ n□□□□□□□□□□□.

□□□ 3□□,n□□□ $a_1, a_2, \ldots, a_n$ □□□□□□□□□□□□□□□□□,□□ n□□□□□□□□ $a_1, a_2, \ldots, a_n$,□□□□□$(a_1, a_2, \ldots, a_n)$□□□□□□□□□□□.□□□□□□

$$(a_1, a_2, \ldots, a_n) = ((a_1, a_2, \ldots, a_{n-1}), a_n). \quad (2)$$

□(2)□□□□□ 1□□□,□□ n□□□□□□□□ $a_1, a_2, \ldots, a_n$,□□ n□□□ $u_1, u_2, \ldots, u_n$,□□

$$u_1 a_1 + u_2 a_2 + \ldots + u_n a_n = (a_1, a_2, \ldots, a_n). \quad (3)$$

□□4 □□ n□□□ $a_1, a_2, \dots, a_n$,□□$(a_1, a_2, \dots, a_n)=1$,□□ $a_1, a_2, \dots, a_n$□□.

n□□□ $a_1, a_2, \dots, a_n$□□□□□□□□□□□□□ $u_1, u_2, \dots, u_n$,□□

$$u_1 a_1 + u_2 a_2 + \dots + u_n a_n = 1.$$

□□5 □ a, b□□□,□□□□ m□□ a□ b□□□□□□,□□

1) $a \mid m$, $b \mid m$;

2) □ $a \mid l$, $b \mid l$,□□□ $m \mid l$.

□□□□,□□□□□□□□□□□□□,□□□□□□□□□□□□□. □□□□□□□□□ a, b,□□□$[a, b]$□□□□□□□□□□□.□□ $a>0$, $b>0$,□

$$[a, b](a, b) = ab. \tag{4}$$

□□□□□□.

□□□□□5,□□□ $n(n>2)$□□□□□□□□□□□□,□□□□□□ □□,□□□□□□□□□□.□□□□□□□□□ $a_1, a_2, \dots, a_n$,□ $[a_1, a_2, \dots, a_n]$□□□□□□□□□□□,□□

$$[a_1, a_2, \dots, a_n] = [[a_1, a_2, \dots, a_{n-1}], a_n].$$

§4 □□□□□□,□□□□□□□□

□□□□□□,□□ K□□□□□□□□ $K[x]$□□□□□□.□□□□ □, $K[x]$□□□□□□□□□□□□□□□.□□□□□□□□□□□□□ $K[x]$□□□.□□□□ Z□□:□□□□□□□□□□□□□□□□□□□ □,□□□□□□□: $K[x]$□□□□□□□□□□□□□□□□□□□"□ □"□□□□□□□□□□□?□□□□1□□□□ p□□□□,□□ p□□ □□□□1□ p□□.□□□□□□,□□□□□□:

□□1 $K[x]$□□□□□□□□□□□□ $p(x)$,□□□□ $K[x]$□□□ □□□□□□□□□ $p(x)$□□□□,□□ $p(x)$□□□ K□□□□□□□□ □□;□□□ $p(x)$□□□□□□.

□□1 $K[x]$□□□□□□□ $p(x)$□□□□□□ $f(x)$□□□□□□□ □□:□□ $p(x) \mid f(x)$,□□ $p(x)$□ $f(x)$□□.

□□ □□$(p(x), f(x))$□ $p(x)$□□□,□ $p(x)$□□□,□□ $(p(x), f(x))$□□□□□□,□□$(p(x), f(x))$□ $p(x)$.□□$(p(x), f(x))=1$,□□ $p(x) \mid (p(x), f(x))$.□□□□□□□□□□□□ $p(x) \mid f(x)$. x

□□ 2 □ $K[x]$□, □□ $p(x)$□□□, □ $p(x)\mid f(x)g(x)$, □ $p(x)\mid f(x)$□ $p(x)\mid g(x)$.

□□ □□ $p(x)\mid f(x)$, □□□□□. □□□ $p(x)\nmid f(x)$. □□ $p(x)$□□□, □□□□□ 1□, $(p(x), f(x))=1$. □□□ $p(x)\mid f(x)g(x)$□□, $p(x)\mid g(x)$. x

□□□□□□□, □□ 2□□□□□: □ $K[x]$□, □□ $p(x)$□□□, □ $p(x)\mid f_1(x)f_2(x)\dots f_s(x)$, □ $p(x)\mid f_j(x)$, □□□□ j□$\{1,2,\dots,s\}$.

□□ 3 $K[x]$□, $p(x)$□□□□□□□ $p(x)$□□□□□□□□□□ $p(x)$□□□□□□□□□□□.

□□ □□□. □□ $p(x)$□□□, □ $p(x)$□□□□□□□□□□□ $p(x)$□□□□, □□ $p(x)$□□□□□□□□□□ $p(x)$□□□□□□□□□□□.

□□□. □□□□, □□ $p(x)$□□, □ $p(x)$□□□ $g(x)$□□ $0<\deg g(x)<\deg p(x)$. □□□□ $h(x)$□$K[x]$□□

$$p(x)=h(x)g(x),$$

□□
$$\deg p(x)=\deg h(x)+\deg g(x).$$

□□□□,
$$0<\deg h(x)<\deg p(x).$$

□□□□□□□□. x

□□□ 3□□□□, $K[x]$□□□□□ **1**□□□□□□□□□□□□□.

□□□ 3□□□□□□□□□:

□□ 1(□□□□□□□□) $K[x]$□□□□□□□□□□□□□ $f(x)$□□□□□□□□□□ K□□□□□□□□□□□□□□. □□□□□□□, □□ $f(x)$□□□□□□□□□:

$$f(x)=p_1(x)p_2(x)\dots p_s(x)=q_1(x)q_2(x)\dots q_t(x),$$

□□□□ $s=t$, □□□□□□□□□□□□

$$p_i(x)\text{□}q_i(x),\quad i=1,2,\dots,s.$$

□□ □□□□□□□□□. □□□□□□□ n□□□□□□.

□□□□□□□□□□□□□, □□ $n=1$□□□□□.

□□□□□□□□ n□□□□□□□□, □□□□ n□□□□ $f(x)$.

□□ $f(x)$□□□□□□□, □□□□□□□. □□ $f(x)$□□□□□□, □□

$$f(x)=f_1(x)f_2(x),$$

□□ $f_i(x)$□$K[x]$, □□ $\deg f_i(x)<\deg f(x)$, $i=1,2$. □□□□□, $f_1(x)$□$f_2(x)$□□□□□□□□ K□□□□□□□□□□□□□□□, □ $f_1(x)$□$f_2(x)$□□□□□□□□□□ $f(x)$□□□□□□.

□□□□□□,□□□□□□.

□□□□□□.□□ $f(x)$ □□□□□□□□□

$$f(x)=p_1(x)p_2(x)\cdots p_s(x)=q_1(x)q_2(x)\cdots q_t(x),\qquad(1)$$

□□ $p_i(x), q_j(x), i=1,\ldots,s, j=1,\ldots,t$ □□□□ K □□□□□□□□.

□□□□□□□□□□□□□□ s □□□□.

□ $s=1$ □, $f(x)=p_1(x)$, □ $f(x)$ □□□□□□□.□□□□□□□□□□□□□□□□□□□□□□□□,□□□ $q_1(x)\mid f(x)$□

$$q_1(x)\,□\,f(x),$$

□□ $f(x)=cq_1(x)$, $c\,□\,K^*$. □□ $t=1$, □ $p_1(x)\,□\,q_1(x)$.

□□□□□□□□□□□□□□□ $s-1$ □□□□□.□□□□□□□□□□□□□ s □□□.

□(1)□, $p_1(x)\mid q_1(x)q_2(x)\cdots q_t(x)$, □□ $p_1(x)$ □□□□□□□□□,□□□

$$p_1(x)\mid q_1(x),$$

□□ $q_1(x)$ □□□□□□□□,□□ $p_1(x)\,□\,q_1(x)$, □

$$p_1(x)=c_1q_1(x),\quad c_1\,□\,K,\quad c_1\,□\,0,\qquad(2)$$

□(2)□□(1)□,□□□□□□ $q_1(x)$, □

$$p_2(x)\cdots p_s(x)=(c_1^{-1}q_2(x))\cdots q_t(x).\qquad(3)$$

□□□□□,□

$$s-1=t-1,\ □\ s=t,\qquad(4)$$

□□□□□□□□□□□□□□

$$p_2(x)\,□\,c_1^{-1}q_2(x),\ p_3(x)\,□\,q_3(x),\ \ldots,\ p_s(x)\,□\,q_s(x).\qquad(5)$$

□□□

$$p_i(x)\,□\,q_i(x),\quad i=1,2,\ldots,s.\qquad(6)$$

□□□□□□□□□,□□□□□. X

□□□□□□□□□□□, $f(x)$ □□□□□□□□□□□ $f(x)$ □□□□□□□□□□□□□□□□□.□□ $f(x)$ □□□□□□□□□□□□□□□□(□□□□□□).

$K[x]$ □□□□□□□□□□□□□□□□□□,□□□□□□□□□□□□□□□□□□□□,□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□.

□□□□ $f(x)$ □□□□□,□□□□□□□□□□□□□□□□□□,□□□□□□□□□□ 1 □□□□,□□□□□□□□□□□□□□□□□□□□,□□ $f(x)$ □□□□□□

$$f(x)=cp_1^{r_1}(x)p_2^{r_2}(x)\cdots p_m^{r_m}(x), \tag{7}$$

□□ c□ $f(x)$□□□□□, $p_1(x), p_2(x), \ldots, p_m(x)$□□□□□□□□□ 1□□□□□□□, $r_1, r_2, \ldots, r_m$ □□□□. □□□□□(7)□□ $f(x)$□□□□□□.

□□□□□□□□, □□□□□□□□□ $f(x)$□ $g(x)$□□□□□□, □□$(f(x), g(x))$□□□□□□□ $f(x)$□ $g(x)$□□□□□□□□□□□□□□□□□□□□□□□□,□□□□□□□□□□□□ $f(x)$□ $g(x)$□□□□□□□□□□□□□□□□. □□

$$f(x)=ap_1^{k_1}(x)\cdots p_l^{k_l}(x)p_{l+1}^{k_{l+1}}(x)\cdots p_m^{k_m}(x), \tag{8}$$

$$g(x)=bp_1^{t_1}(x)\cdots p_l^{t_l}(x)q_{l+1}^{t_{l+1}}(x)\cdots q_s^{t_s}(x), \tag{9}$$

□

$$(f(x), g(x))=p_1^{\min\{k_1,t_1\}}(x)\cdots p_l^{\min\{k_l,t_l\}}(x). \tag{10}$$

□□□□:□□(10)□□□□□□□□□ $d(x)$□ $f(x)$□ $g(x)$□□□□□□. □□ $f(x)$□ $g(x)$□□□□□□□□□□□□ $c(x)$, □ $c(x)$□□□□□□□□□□□□□□□□ $u(x)$, □□ $u(x)|f(x)$□ $u(x)|g(x)$, □□ $u(x)$□□□□ $p_i(x)$, □□ 1□ i□ l; □□ $u(x)$□ $c(x)$□□□□□□□□□□□□□□ $\min\{k_i, t_i\}$. □□□□, $c(x)|d(x)$. □□□□ $d(x)$□ $f(x)$□ $g(x)$□□□□□□. x

□□□□□□□□□□□□□□□□□□□□□□□□, □□□□□□□□□□□□□□□□□□□□□□□□.

□□□□□□□□□□□□□□□□□□□□□□□□□.

□ 1 □□ x^2-2□□□□□□□□□.

□□ □ x^2-2□□□□□ Q□□□, □□□□□□□□□□

$$x^2-2=(x+a)(x+b),\quad a, b□Q. \tag{11}$$

(11)□□□□□□ x^2-2□□□□ R□□□□□□□□□□□. □□□□□□□□ x^2-2□ R□□□□□□□□□□□□

$$x^2-2=(x+2)(x-2). \tag{12}$$

□ R[x]□□□□□□□□□□

$$x+a=x+2 \quad □ \quad x+a=x-2.$$

□□□□ $a=2$□ $a=-2$, □□ a□Q□□. □□ x^2+2□ Q□□□□.

□□□□□ x^2-2□□□□□□□□□□□□□□□□. (□□:□□□□□,□(11)□□□□□□□□□□□□□□□□.)

□□ 7.4

1. □□□□□ x^2+1□□□□□□□□□.

2. □□□□□□,□□□□□□□□□□□□□□□□□□□□□□□□□.

(1) x^4+1; (2) x^4+4.

3. □□:□ $K[x]$□, $g^2(x)\mid f^2(x)$□□□□ $g(x)\mid f(x)$.

4. □□□□□□ 2□□□□□,□□ $p(x)$□ $K[x]$□□□□□□□□□□□□,□□□□□□ $f(x), g(x)$□ $K[x]$,□ $p(x)\mid f(x)g(x)$□□□□ $p(x)\mid f(x)$□□ $p(x)\mid g(x)$,□□ $p(x)$□□□□□□□.

*5. □□:□□ K□□□□□□□□□□□□ $f(x)$□ $K[x]$□□□□□□□□□□□□□□□□□□□□□□□ $g(x)$□ $K[x]$,□□$(f(x), g(x))=1$,□□□□□□□□□ m□□ $f(x)\mid g^m(x)$.

*6. □□:□□ K□□□□□□□□□□□□ $f(x)$□ $K[x]$□□□□□□□□□□□□□□□□□□□□□□□□ $g(x), h(x)$□ $K[x]$,□ $f(x)\mid g(x)h(x)$□□□□ $f(x)\mid g(x)$,□□□□□□□□ m□□ $f(x)\mid h^m(x)$.

7. □ $K[x]$□,□$(f, g_i)=1$, $i=1,2$,□□:

$$(fg_1, g_2)=(g_1, g_2).$$

□□□□□:□□□□□□□□□□□□

□□ 1 □□□□ 1□□□,□□□□□□□□□ 1□□□□,□□□□□,□□□□□□□.

□□□ 1□□□□,□□□□ p□□□□□ a□□□□□□□□□□□:□□$(p, a)=1$,□□ $p\mid a$.

□□□□□,□□□□

□□ 1 □□ p□□□,□□□□□□□□□ a, b,□ $p\mid ab$□□□□ $p\mid a$□□ $p\mid b$.

□□ 1□□□□□:□□□□ p□□□□□□□ $a_1, a_2, \ldots, a_s$□□□ $a_1a_2\ldots a_s$,□ p□□□□□□□□□.

□□ 1□□□□□□□.□□□□□□.

□□ 2(□□□□□□□□□□□) □□□□ 1□□□ a□□□□□□□□□□□□□□□□.□□□□□□□,□□ a□□□□□□□□□

$$a=p_1p_2\ldots p_s=q_1q_2\ldots q_t,$$

□□□□ $s=t$,□□□□□□□□□□□□□

$$p_i = q_i, \quad i=1,\dots,s.$$

□□ □□□□□□□□□,□□□□□□.

□ $a=2$□,2□□□,□□□□□□□□.

□□□□□ a□□□ 1□□□□□,□□□□,□□ $a(a>1)$.□□ a□□□,□□□□□.□□ a□□□,□□□□□□ 1□□□ a□□□ b,c,□□ $a=bc$.

□□□□□,b□ c□□□□□□□□□□□□□,□□ a□□□□□□□□□□□□.

□□□□□□,□□□□□.

□□□□□□§4□□□ 1□□□□□□□□,□□□□□. x

□□ 2□□□□□□□□,□□□□□□□□□□□□□□□□□□□□□□.□□□□□□□□□□,□□□□ 1□□□ a□□□□□□(□□□□□□□□□□)

$$a = p_1^{r_1} p_2^{r_2} \cdots p_m^{r_m}, \tag{1}$$

□□ $p_1, p_2, \dots, p_m$□□□□□□,$r_1, \dots, r_m$□□□□,(1)□□□ a□□□□□□.

□□□□□□□□ 1□□□ a, b□□□□□□

$$a = p_1^{k_1} \cdots p_s^{k_s} p_{s+1}^{k_{s+1}} \cdots p_m^{k_m}, \tag{2}$$

$$b = p_1^{l_1} \cdots p_s^{l_s} q_{s+1}^{l_{s+1}} \cdots q_n^{l_n}, \tag{3}$$

□ $$(a, b) = p_1^{r_1} \cdots p_s^{r_s}, \quad r_i = \min\{k_i, l_i\}, \ i=1,\dots,s, \tag{4}$$

$$[a, b] = p_1^{\delta_1} \cdots p_s^{\delta_s} p_{s+1}^{k_{s+1}} \cdots p_m^{k_m} q_{s+1}^{l_{s+1}} \cdots q_n^{l_n}, \ \delta_i = \max\{k_i, l_i\}, \ i=1,\dots,s. \tag{5}$$

(4)□□(5)□□□□□□□□□□□□.

§5 □□□

□□□□□□□□ $K[x]$□□□□□□□□□□□□□ $f(x)$□□□□□□□□□ K□□□□□□□□□□□□□□.□□ $f(x)$□□□□□□□□□□□□□□□□ 1□,□□□□□□□□□□□□.□□□□□□□□□□□□□□□□□□□□□□.

□□ 1 □ $K[x]$□,□□□□□□ $p(x)$□□□□□ $f(x)$□ k□□□,□□ $p^k(x) \mid f(x)$,□ $p^{k+1}(x) \nmid f(x)$.

□□ $k=0$,□ $p(x) \nmid f(x)$,□□ $p(x)$□□ $f(x)$□□□;□□ $k=1$,□

$p(x)$□□ $f(x)$□□□□;□□ $k>1$,□ $p(x)$□□ $f(x)$□□□□.

□□,□□ $f(x)$□□□□□□□

$$f(x)=cp_1^{r_1}(x)p_2^{r_2}(x)\cdots p_m^{r_m}(x),$$

□ $p_1(x),p_2(x),\ldots,p_m(x)$□□□ $f(x)$□ r_1□,r_2□,…,r_m□□□.□□ $r_i=1$□□□□□□□□□□□□;□□ $r_i>1$□□□□□□□□□□□□.□□ $f(x)$□□□□□□□□□□□□□□□□□ 1□□□□□□□ $f(x)$□□□□□□□□.□□□□□□□□□□□□□□□□?□□□□□□□□□□□□□□□□□□□□□□,□□□□□□□□□□□□□□□□□□□□□□□□□□.

□□□□□□□□□□,□□□□□□□□.

□ $f(x)=(x+1)^3$□ $R[x]$,□□ $f(x)$□□□□.□□□□□ $f(x)$□□□□□□□□□□□□□□□□,□□□ $f(x)$□□□□□,□ $f'(x)=3(x+1)^2$.□□ $(f(x),f'(x))=(x+1)^2$.□□□□□□□□□,□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□.□□□□□□□□□□□□□□□□ K□□□□□□□□□□,□□□□□□□□□□□□□□□ x□□□,□□□□□□□□□□□,□□□□□□□□□□□□□□□□□□□□□□□□□,□□□□□□□□□ K□□□□□□□□□□□□□,□□□□□□□□□□□□□□□□□□□□□□□□□□.

□□ **2** □□ $K[x]$□□□□□

$$f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0,$$

□□□ $K[x]$□□□□□

$$na_nx^{n-1}+(n-1)a_{n-1}x^{n-2}+\cdots+a_1,$$

□□ $f(x)$□□□(□□□□□),□□ $f'(x)$.

$f'(x)$□□□□□ $f(x)$□□□□□,□□ $f''(x)$;$f''(x)$□□□□□ $f(x)$□□□□□,□□ $f'''(x)$,□□.$f(x)$□ k□□□□□ $f^{(k)}(x)$.

□□□ 2□□□□,□□ n□□□□□□□□□□□ $n-1$□□□□;□□ n□□□□ K□□□□□□;□□ $n+1$□□□□□.□□□□□□□□□□□□.

□□□□ 2,□□□□□□,□□□□ $K[x]$□□□□□□□□□□□□:

$$[f(x)+g(x)]'=f'(x)+g'(x),$$

$$[cf(x)]'=cf'(x),\ c\in K,$$

$$[f(x)g(x)]'=f'(x)g(x)+f(x)g'(x),$$

$$[f^m(x)]'=mf^{m-1}(x)f'(x).$$

□□□□□□□□□.□□□□□□ $x+1$□ $f(x)=(x+1)^3$□ 3□□

□. □□□□□ 2 □□□□□□□□, $f'(x)=3(x+1)^2$, □□ $x+1$ □ $f'(x)$ □ 2 □□□. □□□□□□□□□□□, □

□□ **1** □ $K[x]$ □, □□□□□□□□ $p(x)$ □ $f(x)$ □□□ $k(k\geq 1)$ □□□, □ $p(x)$ □ $f(x)$ □□□ $f'(x)$ □□□ $k-1$ □□□. □□□, □□□ $f(x)$ □□□□□□ $f(x)$ □□□ $f'(x)$ □□□.

□□ □□ $p(x)$ □ $f(x)$ □ k □□□, □□□□ $g(x)$ □ $K[x]$, □

$$f(x)=p^k(x)g(x),\quad p(x)\nmid g(x).$$

□ $f(x)$ □□□, □

$$f'(x)=p^{k-1}(x)[kp'(x)g(x)+p(x)g'(x)].$$

□□ $p(x)\nmid kp'(x)$, $p(x)\nmid g(x)$, □□ $p(x)$ □□□□□□□, □□ $p(x)\nmid kp'(x)g(x)$. □□ $p(x)\mid p(x)g'(x)$, □□ $p(x)$ □□□□□□□□□□□□□□□. □□ $p(x)$ □ $f'(x)$ □ $k-1$ □□□. x

□□ **2** □ $K[x]$ □, □□□□□□ $p(x)$ □ $f(x)$ □□□□□□□□□□□ $p(x)$ □ $f(x)$ □ $f'(x)$ □□□□.

□□ □□□. □□□□□□□ $p(x)$ □ $f(x)$ □ k □□□, □□ $k>1$, □□□ 1□, $p(x)$ □ $f'(x)$ □ $k-1$ □□□, □□ $p(x)$ □ $f(x)$ □ $f'(x)$ □□□□.

□□□. □□□□□□□ $p(x)$ □ $f(x)$ □ $f'(x)$ □□□□. □□□ 1□, $p(x)$ □□ $f(x)$ □□□□, □□ $p(x)$ □ $f(x)$ □□□□. x

□□□ 2□□□□: $K[x]$ □□□□□□□□□ $f(x)$ □□□□□□□□□□□, $f(x)$ □ $f'(x)$ □□□□□□□□□□. □□□□□□□□ 3:

□□ **3** $K[x]$ □□□□□□□□□ $f(x)$ □□□□□□□□□□□□□ $f(x)$ □□□□□ $f'(x)$ □□. x

□□ 3□□, □□□□□□□□ $f(x)$ □□□□□□, □□□□□ $(f(x), f'(x))$, □□□□□□□□□□□□□: □□□□□, □□□□□□□□□□——□□□□□□□□□□□□□□□□□□. □□□□, □□□□□□□□, □□□□□□□□□□□□, □□□□□□□□□□□□, □□□□□□□□□:

□□ **4** □□□ F □□□□ K, □□ $f(x)\in K[x]$, $f(x)$ □ $K[x]$ □□□□□□□□□□□□□□□ $f(x)$ □□ $F[x]$ □□□□□□, $f(x)$ □ $F[x]$ □□□□□□. □, $f(x)$ □□□□□□□□□□□□□□□□. x

□□□□□, □□□□□ $f(x)$ □□□, □□□□□□□□□□□ $g(x)$, □□□□□□, □□□□□□, □□ $f(x)$ □□□□□□□□□□□. □□□□□□□□□□ $g(x)$ □□□.

□ $K[x]$ □□□□□ $f(x)$ □□□□□□□

$$f(x)=cp_1^{r_1}(x)p_2^{r_2}(x)\cdots p_m^{r_m}(x),$$

□□□ 1□

$$f'(x)=p_1^{r_1-1}(x)p_2^{r_2-1}(x)\cdots p_m^{r_m-1}(x)h(x),$$

□□ $h(x)$□□□□□ $p(x)$□□, $i=1,\ldots,m$□□

$$(f(x),f'(x))=p_1^{r_1-1}(x)p_2^{r_2-1}(x)\cdots p_m^{r_m-1}(x).$$

□□□$(f(x),f'(x))$□ $f(x)$□□□□□

$$cp_1(x)p_2(x)\cdots p_m(x),$$

□□□□ $g(x)$,□□□□□□□□□□□□□□□□ $g(x)$,□□ $f(x)$□□□□□□□□□□□□(□□□□).

□□□□□□□□□ $f(x)$□□□□□□□□□□□:□□□□□□□□□$(f(x),f'(x))$,□□□ $f(x)$□$(f(x),f'(x))$□□□□□,□□□□ $g(x)$□□□□□□□□□□□□□□.

□□ $f(x)$□□□□□□□□□□□□□□.□□,□□□ $f(x)$□□□□□□□□,□□□□□□□□□□□□□□□□□□□□□□□□ $g(x)$,□□$f(x)$□□□□□□□□□□□□(□□□□),□□□ $g(x)$□□□□□ $f(x)$□□□,□□ $g(x)$□□□□□□□□□□□□□□.□□□□□□□ $g(x)$□□□□□□□□ $p(x)$,□□□□□□□□□□ $p(x)$□ $f(x)$□□□□.□□,□□□□□□□□□□□□□□□□ $f(x)$□□(□□□□□□□,□□□□□□□□□.□□□,□□□□□□□□□□□□□□□□□□□),□□□□□□□□□□□□□□□□□□□□□□□□,□□□□□□□□□ $f(x)$□□□□□□□□□,□□□□□□□□□□□□□ $g(x)$,□ $g(x)$□ $f(x)$□□□□□□□(□□□□).

□ **1** □□:$Q[x]$□□□□□

$$f(x)=1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}$$

□□□□□.

□□ □ $f(x)$□□□□

$$f'(x)=1+x+\cdots+\frac{x^{n-1}}{(n-1)!},$$

□□ $f(x)=f'(x)+\frac{x^n}{n!}$.□□□ 7.3□□ 7□□□□□

$$(f(x),f'(x))=\left(f'(x)+\frac{1}{n!}x^n,\ f'(x)\right)=\left(\frac{x^n}{n!},\ f'(x)\right).$$

□□ $\frac{x^n}{n!}$□□□□□□□□ x(□□□□),□ $x\nmid f'(x)$,□□

$$\frac{x^n}{n!}, \quad f'(x) = 1,$$

□□$(f(x), f'(x))=1$. □□ $f(x)$□□□□□. x

□ □ 7.5

1. □□□□□□□□□□□□□□□□:

(1) $f(x)=x^3-3x^2+4$;

(2) $f(x)=x^3+2x^2-11x-12$.

2. a, b□□□□□□□□,□□□□□□□□□□□□□□:

$$f(x)=x^3+2ax+b.$$

3. □□□□: $K[x]$□,□□□□□□□ $p(x)$□ $f(x)$□□□ $f'(x)$□ $k-1$□□□ (k□2),□□ $p(x)$□□ $f(x)$□ k□□□.

4. □□: $K[x]$□,□□□□□□□ $p(x)$□ $f(x)$□□□ $f'(x)$□ $k-1$□□□(k□1),□□ $p(x)$□ $f(x)$□□□,□ $p(x)$□ $f(x)$□ k□□□.

5. □□: $K[x]$□,□□□□□□ $p(x)$□ $f(x)$□ k□□□(k□1)□□□□□□□□: $p(x)$□ $f(x), f'(x), \ldots, f^{(k-1)}(x)$□□□,□□□ $f^{(k)}(x)$□□□.

6. □ $Q[x]$□□□□□□□□□□□□□ $g(x)$,□□□ $f(x)$□□□□□□□□□□□□(□□□□);□□□ $f(x)$□□□□□□:

$$f(x)=x^5-3x^4+2x^3+2x^2-3x+1.$$

*7. □□: $K[x]$□□□ n□(n□1)□□□ $f(x)$□□□□□□□□□□□□□□□□□□□□□□□□ n□□□□.

§6 □□□□□,□□□□□□□□□□□

□□□□□□□□□□□, $K[x]$□□□□□□□□□□□□□□□□□□□□□□ K□□□□□□□□□□□□□□. □□□□,□□□□□□□□□□□□□□□□,□□□□□□□□ $K[x]$□□□□□□□□□□? □□□□□□, $K[x]$□□□□□□□□□□□□□□□. □□□□□□□□□□□, $K[x]$□□□□□□□□ 1□□□□□□□? □□,□ $K[x]$□,□□ $p(x)$□□□□□ 1□□□□□□□,□ $p(x)$□□□□□□. □□□□□□□,□□□□□□ $K[x]$□□□□□ $f(x)$□□□□□□□□□□□□□. □□□□□□□□□□□□ $f(x)$,□□□□□□.

□□ 1(□□□□) □ $K[x]$□,□ $x-a$□□ $f(x)$□□□□□□ $f(a)$.

□□ □□□□□,□

$$f(x)=h(x)(x-a)+r(x),\quad \deg r(x)<\deg(x-a).$$

□ $r(x)$□0,□ $\deg r(x)=0$,□ $r(x)=r$,□□ r□ K^*.□ $r(x)=0$,□□□□ $r(x)$□ K□□□ 0□□.□□□□□

$$f(x)=h(x)(x-a)+r,\qquad r\text{□}K. \tag{1}$$

x□ a□□,□(1)□□,$f(a)=r$.□□□ $x-a$□□ $f(x)$□□□□□□ $f(a)$.

x

□□ **2** □ $K[x]$□,$x-a$□□ $f(x)$□□□□

$$f(a)=0.$$

x

□□□ 2□□□□□□□□□□□□,□□□□□□□□□□.

□□ **1** □ K□□□□□,□ R□□□□□□□□□□□,□ R□□□□ K□□□□□.□□ $K[x]$□□□□□□ $f(x)$,□□ R□□□□□□ c□□ $f(c)=0$,□□ c□ $f(x)$□ R□□□□□.

$f(x)$□□□□□□□□□□□.□□□,□□□□□□□□□□□□□□□□□□;□□□□□□□□□□□□□□□□□□□□.

□□□ 1□□□ 2□□□□□□□□□□:

□□ **3(Bezout □□)** $K[x]$□,$x-a$□□ $f(x)$□□□□ a□ $f(x)$□ K□□□□□.

x

□□□ 3□□,$K[x]$□□□□□ $f(x)$□□□□□□□□□□□□□ $f(x)$□ K□□□.

□□□□□□□□□□□,□□ $K[x]$□□□□□ K□□□,□□□□□□□□□□:a□ K□□ $f(x)$□ $K[x]$□□□ k□□,□□ $(x-a)$□ $f(x)$□ k□□□.□ $k=1$□,a□□□□;□ $k>1$□,a□□□□.

□□□□□□□□□□□,□□□□□□ $K[x]$□□□□□ $f(x)$□ K□□□□□□□□□□:

□□ **4** $K[x]$□□ n□(n□0)□□□□ K□□□□ n□□(□□□□□□□).

□□ □□□□□□□□,□□□□□□.

□ $f(x)$□□□ n□□□□,$n>0$.□ $f(x)$□□□□□□□□□□□□□.□□□ 3□□□□□□□□□□□□□,$f(x)$□ K□□□□□□(□□□□□□□)□□□□□□□□□□□□□(□□□□□□□□),□□□□□□□□□ $f(x)$□□□ n.

x

□□□ 4□□□□□□□□□□:

□□ **5** □ $K[x]$□□□□□□ $f(x)$□ $g(x)$□□□□□□□ n.□□ K□□ $n+1$□□□□□ $a_1,a_2,\ldots,a_{n+1}$,□□ $f(a_i)=g(a_i)$, $i=1,2,\ldots,n+1$,

□ $f(x) = g(x)$.

□□ □ $h(x) = f(x) - g(x)$, □□ $h(x)$ □ 0, □

$$\deg h \,□\, \max\{\deg f, \deg g\} \,□\, n.$$

□□

$$h(a_i) = f(a_i) - g(a_i) = 0, \quad i = 1, 2, \dots, n+1,$$

□□ $h(x)$ □ K □□□□ $n+1$ □□□□□. □□□□ 4 □□. □□ $h(x) = 0$. □ □ $f(x) = g(x)$.

□□□□□□□□□□□□□ 5 □□□.

□□□□□□□□□, □□□□□□□□□□□□□, □□□□□□□□□ □□□□□□□□.

□ $f(x)$ □ $K[x]$, □□ K □□□□□□ a, x □ a □□□, $f(a)$ □ K. □□ $K[x]$ □□□□□□ $f(x)$ □□□ K □ K □□□□□(□ K □□□□□□), □ □ f □□, □

$$f: K \to K$$
$$a \longmapsto f(a).$$

□□□ $K[x]$ □□□□□□□ K □□□□□ K □□□□□□□.

□ K □□□ R □, □□□□□□□□□□□□□□□□□□□□□□ □□□.

□□□□□, $K[x]$ □□□□□□□□□□□□ K □□□□□□. □ □□□: $K[x]$ □□□□□□□□□ $f(x)$ □ $g(x)$, □□□□□ K □□□□ □□□ f □ g □□□□□? □□□□□□, □□□□

□□ **6** □□ K □□□□□□□ $f(x)$ □ $g(x)$ □□□□□, □□□□□□ K □□□□□□□ f □ g □□□□.

□□ □□□□□. □ K □□□□□□□ f □ g □□, □□□□ a □ K, □ $f(a) = g(a)$. □□ K □□□, □□□□□□□□, □□□□□ 5 □, $f(x) = g(x)$. □□□□ $f(x)$ □ $g(x)$ □□.

□□, □□ 6 □□□□□□□ K □□□□□□□.

□□□□□ K □□□□□□□□□□□□□□ K_{pol}. □ $K[x]$ □□□□□ $f(x)$ □□□□□□□ K □□□□ f, □□ $K[x]$ □ K_{pol} □□□□□, □□ σ. □ □□□□□. □□ 6 □□□□□□□□□, □□□□□□.

□□□ K_{pol} □□□□□□□□□□: □ f, g □ K_{pol}, □□

$$(f+g)(a) \overset{\mathrm{def}}{=} f(a) + g(a), \quad \forall a \in K,$$

$$(fg)(a) \overset{\mathrm{def}}{=} f(a)g(a), \quad \forall a \in K.$$

□□□□, $f+g$, fg□ K_{pol} (□ f, g□□□□□ $f(x)$, $g(x)$□□□□□,□

$$h(x)=f(x)+g(x),$$

□□□□ a□ K,□

$$h(a)=f(a)+g(a)=(f+g)(a),$$

□□ $h=f+g$.□□□ $f+g$□□□□ $f(x)+g(x)$□□□□□.□□□□ fg □□□□ $f(x)g(x)$□□□□□),□□ K_{pol}□□□□,□□□ K□□□□□□ □□.□□□□□□□□□□□□□□□ 0,□□□□□;□□□□□□□□□ □ 1□□□□□ 1,□□□□□□□□,□□ K□□□□□ a□□ 1.□□□□ K_{pol}□□□□□□.

□ $K[x]$□□ K_{pol}□□□□□:

□□ 7 □ K□□□,□ $K[x]$□□□□□ $f(x)$□□□□□□□ K□□ □□ f,□□□□□ σ□□ $K[x]$□□ K_{pol}□□□,□□ σ□□□□□□□□□.

□□ □□□□ σ□□□.□□□ σ□□□□.

□ $f(x)$, $g(x)$□ $K[x]$,□□□□ $f(x)+g(x)$□□□□□□ $f+g$, $f(x)g(x)$□□□□□□ fg.□□

$$\sigma(f(x)+g(x))=f+g=\sigma(f(x))+\sigma(g(x)),$$

$$\sigma(f(x)g(x))=fg=\sigma(f(x))\sigma(g(x)).$$

□□□□ σ□□□□,□□□□□□. x

□□ 2 □ R□ R□□□□,□□□□□ R□ R□□□□□□ σ,□□□□ □□□□□□,□□□□ a, b□ R,□

$$\sigma(a+b)=\sigma(a)+\sigma(b),$$

$$\sigma(ab)=\sigma(a)\sigma(b),$$

□□□ σ□□ R□ R□□□□□□□,□□□□ R□ R□□□□□,□□ Rṻ R□.

□□ 7□□,□□ K□□□□□□□□ $K[x]$□□□ K□□□□□□□□ K_{pol}□□□□.□□□□,□□□□□□□□□□□□□□,□□□□□□□ □□□□□□□.□□□□□□□□□□□□□□□□□□,□□□□□□□ □□□,□□□□□□□□□□□□□□.

□□□□□□□□□□□□□□□□□□ $K[x]$□□□□□□ K□□□: a□ K□ $K[x]$□□□□ $f(x)$□ K□□□□□□□ K□□□□□□□ f□ a □□□□□ $f(a)=0$.□□□□□□□□□□□□□□□□□ $K[x]$□□□□ □□ K□□□,□□□□ K□□□□ R□□□□ C□□□.□□,□□□□□ □ $f(x)$□□□□□□□□□□□ $y=f(x)$□□□□ x□□□□□□□□.□□ □□□□□,□□□□□□□□□□□,□□□□□□□□□□□□□:□□□

$[a, b]$□, □□ $f(a)$□ $f(b)$□□□□□□. □□□□ $f(a)<0$, □ $f(b)>0$, □□□ c□(a, b), □□ $f(c)=0$, □□□□, □□□(a, b)□□□□□□□ $f(x)$ □□□□.

□□□□□□□□□□□□□□□□□□□□□□□.

□□ 8(□□□□□□) □□□□□□□□□□□□□□□□□□□□□□□□□□□□.

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□.

*□□ 9(Liouville □□) □□□ $f(z)$□□□□ C□□□, □□□□, □ $f(z)$□□□□□□□. x

* Liouville□□□□□□□□□□□□□□□□□□□. □□□□□□□, □□ $f(z)$□□□□ C□□□□□□□: $f(z)$□□□□ C□□□□□□□□□□. □□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□. □□, □□□ $f(z)$, $g(z)$□□□□ C□□□, □□□□ C□□□□ z, □ $g(z)$□0, □ $f(z)/g(z)$□□□□ C□□□, □□

$$\left(\frac{f(z)}{g(z)}\right)' = \frac{f'(z)g(z) - f(z)g'(z)}{g^2(z)}.$$

*□□□□□□□□□ □□□□ n□$(n>0)$□□□□□ $f(x)$□□□□□□□□, □□□□□□ C□□□□ z, □ $f(z)$□0. □□□□□□□□□□□□□ C□□□□□, □□□□

$$\varphi(z) = \frac{1}{f(z)}$$

□□□□□□□. □□ $z\to$□□, $\varphi(z)\to 0$, □□□□□□□ R□ M_1, □□□ $|z|>R$□, $|\varphi(z)|$□ M_1. □□$|z|$□ R□, □□□□ $\varphi(z)$□$|z|$□ R□□□ M_2. □□□□□ z□C, □

$$|\varphi(z)| \square \max\{M_1, M_2\},$$

□ Liouville□□□, $\varphi(z)$□□□□□□□, □□ $f(z)$□□□□□□□, □□□□□ $f(x)$□□□□□□□□. x

□□ 8□□□"□□□□□□□"□□□□ 19□□□□□□□□□□□□□□□□□□. □□□, □□ 8□□□□□□□□□□□□□(□□□□□□□). □□□□□□□□□□□□□□□(Gauss)□ 1799□□□□, □□□□□□□□□□. Jordan, Weyl□□□□□□□.

□□□□□□□□□□□(□ Bezout□□), □□□□□□□□□□□□□:

□□ 10 □□□□□□□□□□□□□□, □□□□□□□□□□□□.

□□□□, □□□□□□□□□□ 1□□□□□□□□□. □□□□, □□

[illegible].[illegible]:

[illegible] 11([illegible]) [illegible].

X

[illegible],[illegible] $f(x)$ [illegible]

$$f(x)=a(x-c_1)^{r_1}(x-c_2)^{r_2}\cdots(x-c_m)^{r_m}, \tag{2}$$

[illegible] a [illegible] $f(x)$ [illegible], $c_1, c_2, \ldots, c_m$ [illegible], $r_1, r_2, \ldots, r_m$ [illegible],[illegible](2)[illegible]

[illegible] 12 [illegible] n [illegible](n [illegible] 0)[illegible] n [illegible]([illegible]).

X

[illegible] K [illegible] $f(x)$ [illegible] $f(x)$ [illegible].

[illegible] $f(x)$ [illegible] $K[x]$ [illegible] 1 [illegible] n [illegible](n [illegible] 1)[illegible],[illegible] $f(x)$ [illegible],[illegible] $f(x)$ [illegible] n [illegible] $c_1, c_2, \ldots, c_n$ ([illegible]),[illegible] $f(x)$ [illegible]

$$f(x)=(x-c_1)(x-c_2)\cdots(x-c_n). \tag{3}$$

[illegible] $f(x)=x^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$, a_i [illegible] K, $i=0,1,\ldots,n-1$,

[illegible](3)[illegible],[illegible](3)[illegible]

$$\begin{aligned}
a_{n-1}&=-(c_1+c_2+\cdots+c_n),\\
&\cdots\cdots\cdots\cdots\\
a_{n-k}&=(-1)^k\sum_{1\leq i_1<i_2<\cdots<i_k\leq n}c_{i_1}c_{i_2}\cdots c_{i_k},\\
&\cdots\cdots\cdots\cdots\\
a_0&=(-1)^n c_1c_2\cdots c_n.
\end{aligned} \tag{4}$$

[illegible] $f(x)$ [illegible],[illegible] Vieta [illegible].[illegible] $c_1, c_2, \ldots, c_n$ [illegible],[illegible] $a_{n-1}, a_{n-2}, \ldots, a_0$ [illegible] K [illegible].

[illegible] $f(x)$ [illegible] a_n [illegible] 1,[illegible](4)[illegible] $\frac{a}{a_n}$ [illegible].

[illegible] 7.6

1. [illegible] $f(x)=x^5+7x^4+19x^3+26x^2+20x+8$ [illegible] $Q[x]$,[illegible] -2 [illegible] $f(x)$ [illegible],[illegible],[illegible]?

([illegible]:[illegible] $x+2$ [illegible] $f(x)$.)

2. □ $f(x)=2x^3-7x^2+4x+a$ □ $Q[x]$，□ a □，□ $f(x)$ □□□，□□□□□□□□□ □□□□.

(□□：c □ Q □ $f(x)$ □□□ � $x-c$ □ $f(x)$ □□□□ � $x-c$ □ $f(x)$ □ $f'(x)$ □ □□□ � $x-c$ □ $(f(x), f'(x))$ □□□.)

3. □□□□□□□□□□□□□□□□□□□□：

$f(x)=x^3-x^2-x-2$； $g(x)=x^4-2x^3+2x^2-3x-2$.

(□□：a □ C □ $f(x)$ □ $g(x)$ □□□ � $x-a$ □ $f(x)$ □ $g(x)$ □□□□ � $x-a$ □ $(f(x), g(x))$ □□□.)

4. □ $f(x)=x^4-5x^3+ax^2+bx+9$ □ $Q[x]$，□□ 3 □ $f(x)$ □□□□，□ a, b.

5. □□：□ $K[x]$ □，□□ $(x+1) \mid f(x^{2n+1})$，□ $(x^{2n+1}+1) \mid f(x^{2n+1})$.

6. □ $f_1(x), f_2(x)$ □ $Q[x]$，□□：□□

$$(x^2+x+1) \mid f_1(x^3)+xf_2(x^3),$$

□ 1 □ $f_i(x)$ □□，$i=1,2$.

7. □ $C[x]$ □□□□□ $f(x)$ □ 0，□□ $f(x) \mid f(x^m)$，□□ m □□□□□ 1 □□□.□□ $f(x)$ □ C □□□□□□□□□□□.

8. □□□□□□ x^n-1 □□□□□.

*9. □ K □□□□，□ R □（□□□□□）K □□□□□□.□ a □ R，□ J_a □ $\{0\}$，□□ J_a □□ $K[x]$ □□□□□□□□□，□□□□□□ R □□□ a，□

$$J_a=\{f(x) \in K[x] \mid f(a)=0\}.$$

□□：(1) J_a □□□□□□□□□□ 1 □□□ $m(x)$，□□ J_a □□□□□□□□ $m(x)$ □□□；

(2) □□ R □□□□□，□ $m(x)$ □ $K[x]$ □□□□.

10. □□：□□ K □□□□□□□□□□□□□□□□□□□.

11. □ A □□□ K □□ n □□□，□□：A □□□□□□ n □□□□□□□ $\operatorname{tr}(A)$，n □□□□□□□□ $|A|$.

12. □ $p(x)$ □□□ K □□□□□□□□□，□□：□□ $K[x]$ □□□□□ $g(x)$ □ $p(x)$ □□□□□□□，□□ $K[x]$ □ $p(x) \mid g(x)$.

§ 7 □□□□□□□□□□□

□□□□□□□□□□□□□□□□□□□□□□.□□□□□□□□□□□□ $a+bi$ □□□，□□ a, b □□□，□□□□□□□□□□□□□□□□□□□□□□□.

□□ 1 □ $f(x)$ □□□□□□□，□□ c □ $f(x)$ □□□□□，□ $\bar{c}$ □□□□□□ $\bar{c}$ □□ $f(x)$ □□□□□.

□□ □

$$f(x)=a_n x^n+a_{n-1}x^{n-1}+\dots+a_1x+a_0,\ a_i\in R,\ i=0,1,\dots,n,$$

[illegible] c [illegible] $f(x)$ [illegible]

$$f(c)=a_nc^n+a_{n-1}c^{n-1}+\dots+a_1c+a_0=0. \qquad (1)$$

[illegible] (1) [illegible]

$$a_n\bar c^n+a_{n-1}\bar c^{n-1}+\dots+a_1\bar c+a_0=0, \qquad (2)$$

[illegible] $f(\bar c)=0$. [illegible] $\bar c$ [illegible] $f(x)$ [illegible]. ×

[illegible] 1 [illegible]

[illegible] **2** [illegible]

[illegible] $f(x)$ [illegible] $R[x]$ [illegible] $f(x)$ [illegible] $f(x)$ [illegible] c. [illegible] c [illegible] $f(x)$ [illegible] $R[x]$ [illegible] $x-c$. [illegible] $f(x)$ [illegible]

$$(x-c)\mid f(x),$$

[illegible] $f(x)=a(x-c)$, a [illegible]. [illegible] $f(x)$ [illegible].

[illegible] c [illegible] 1, $\bar c$ [illegible] $f(x)$ [illegible]. [illegible] $\bar c$ [illegible] c, [illegible] $(x-c, x-\bar c)=1$. [illegible] $C[x]$ [illegible]

$$(x-c)\mid f(x),\quad (x-\bar c)\mid f(x),$$

[illegible] 2 [illegible]

$$(x-c)(x-\bar c)\mid f(x).$$

[illegible]

$$(x-c)(x-\bar c)=x^2-(c+\bar c)x+c\bar c,$$

[illegible] $c+\bar c$ [illegible] $c\bar c$ [illegible]. [illegible] $C[x]$ [illegible]

$$(x^2-(c+\bar c)x+c\bar c)\mid f(x),$$

[illegible] $R[x]$ [illegible]

$$(x^2-(c+\bar c)x+c\bar c)\mid f(x).$$

[illegible] $f(x)$ [illegible] $R[x]$ [illegible]

$$f(x)=a(x^2-(c+\bar c)x+c\bar c),$$

a [illegible]. [illegible] $f(x)$ [illegible].

[illegible] $f(x)=ax^2+bx+c$ [illegible]

- $f(x)$ [illegible] $R[x]$ [illegible]
- $f(x)$ [illegible]
- $b^2-4ac<0$. ×

[illegible] 2 [illegible]

[illegible] **3(** [illegible] **)** [illegible]

[illegible]

□. x

□□,□□□□□□□□□□□□□

$$f(x)=a(x-c_1)^{r_1}\cdots(x-c_s)^{r_s}$$
$$\cdot(x^2+p_1x+q_1)^{k_1}\cdots(x^2+p_tx+q_t)^{k_t},\qquad (3)$$

□□ a□ $f(x)$□□□□□; $c_1,\dots,c_s$ □□□□□; $p_1,q_1;p_2,q_2;\dots;p_t,q_t$ □□□□□□□,□□□□ $p_i^2-4q_i<0,\ i=1,\dots,t;\ r_1,\dots,r_s,\ k_1,\dots,k_t$ □□□□□□.

□□□□□□□ $f(x)$□□□□□□(3)□□,□□□□ b□ $f(x)$□□□□□,□ b□□□□□□b□□ $f(x)$□□□□□,□□□□□□□□□(□□□ $f(x)$□□□□□□□□).□□□□□□□,□□□□□□□□□□□□□□□.

□ □ 7.7

1. □□:□□□□□□□□□□□□□□.

2. □□□□ x^n-1□□□□□□□□□□.

3. □ A□□□□□ n□□□□□,□□□□□□ $|\lambda I-A|$□□ $f(\lambda)$.□□:(1) $f(\lambda)$□□□□□□□□□;

(2) □□ A□□,□ $f(\lambda)$□□□□□□□□□□.

*4. □□:□□ n□□□ A□ B□□□,□□□□□□□□□□□□□□□.

§8 □□□□□□□□□□□□

□□□□□□□□□□□□□□□□□□□□?□□□□□□□□□□□□□□□□□□?□□□□□□□□□□□□□□□□□□□□□□□□□.

□□□§6□□□□□□,□ $K[x]$□,□□□□□□□□ 1□□□□ $p(x)$□□□,□ $p(x)$□□□□□□,□□ $p(x)$□ K□□□□,□□□□□□□□□ $\mathbf{Q}[x]$□□□□□□□□□□.□□□□□□ $\mathbf{Q}[x]$□□□□□ 1□□□□ $f(x)$□□□□□□□?□□ $f(x)$□□□□□□□□ $f(x)$□ $\mathbf{Q}[x]$□□□□□□□□□□.□□□□□□□□ $f(x)$□ $\mathbf{Q}[x]$□□□□□□□□□□□□□□.□□□□□□□□□.

$$f(x)=\frac{1}{2}x^4+\frac{1}{3}x^3-2x+1$$

$$=\frac{1}{6}(3x^4+2x^3-12x+6),$$

□ $g(x)=3x^4+2x^3-12x+6$, □ $f(x)$□ $g(x)$. □□ $g(x)$□□□□□□□,
□□□□□□□□□□□□□□□□ ±1. □□□□□, □□□□□□:

□□ **1** □□□□□□□□□□□

$$g(x)=b_nx^n+\dots+b_1x+b_0,$$

□□□□□□□□□□□□□□□□ ±1, □□ $g(x)$□□□□□□□□.

□□□□□□□□, □□□□□□□□□□□□□□ $f(x)$□□□□□□ □□□ $g(x)$□□. □□□, □□□□□□ $f(x)$□□□□□□□□□□□□□ □ m, □□□□ $\frac{1}{m}$; □□□□□□□□□□□□□□□□□□□□□□, □□ □□□□□□□□□□□□□□□□□□.

□□□□□□□□□□□□□□ $f(x)$, □□□□□□□□□□□□□? □□□□□ 1□□□□□□□□□.

□□ **1** □□□□□□□ $g(x)$□ $h(x)$□ $Q[x]$□□□□□□□ $g(x)=\pm h(x)$.

□□ □□□□□□□. □□□□□□.

□ $g(x)$□ $h(x)$□ $Q[x]$□□□, □□□ r□ Q, □□ $g(x)=rh(x)$. □ $g(x)=\sum_{i=0}^{n}b_ix^i$, $h(x)=\sum_{i=0}^{n}c_ix^i$, □□ b_i, c_i□ Z, $i=0,1,\dots,n$. □□ r□ ±1, □ $r=\frac{q}{p}$, □□ $(p,q)=1$. □□ p,q□□□□□□□□ ±1, □□□ p□ ±1. □□ $pg(x)=qh(x)$. □□ i□□□□□, $pb_i=qc_i$. □□ $p\mid qc_i$. □□ $(p,q)=1$, □□ $p\mid c_i$, $i=0,1,2,\dots,n$, □□ $h(x)$□□□□□□□□. □□ $r=\pm1$, □ $g(x)=\pm h(x)$. x

□□ 1□□□□, □□□□□□□□□□□□□□ $f(x)$, □□□ $Q[x]$□ □□□□□□□□□□□□□, □□□□□□□□. □□□□□□□□□ □□□ $Q[x]$□□□□□□□□. □□□□□□□□□□□□□□□□□□, □

□□ **2(□□ (Gauss)□□)** □□□□□□□□□□□□□□□□□.

□□ □

$$f(x)=a_nx^n+\dots+a_1x+a_0,$$

$$g(x)=b_mx^m+\dots+b_1x+b_0$$

□□□□□□□□. □

$$h(x)=f(x)g(x)=c_{n+m}x^{n+m}+\dots+c_1x+c_0,$$

□□ $c_s = \sum_{i+j=s} a_i b_j$, $s=0,1,\ldots,n+m$

□□ $h(x)$□□□□□□□, □□□□□□ p, □□ $p|c_s$, $s=0,1,2,\ldots,n+m$. □□ $f(x)$□□□□, □□ p□□□□□ $f(x)$□□□□□□□. □□ □□ $k(0□k□n)$□□

$$p|a_0,\ p|a_1,\ \ldots,\ p|a_{k-1},\ p\nmid a_k. \tag{1}$$

□□, □□ $l(0□l□m)$□□

$$p|b_0,\ p|b_1,\ \ldots,\ p|b_{l-1},\ p\nmid b_l. \tag{2}$$

□□ $h(x)$□ $k+l$□□□□□:

$$c_{k+l} = a_{k+l}b_0 + a_{k+l-1}b_1 + \ldots + a_{k+1}b_{l-1}$$
$$+ a_k b_l + a_{k-1}b_{l+1} + \ldots + a_0 b_{k+l}, \tag{3}$$

□(1),(2)□□□, $p\nmid c_{k+l}$(□□□ $p\nmid a_k$ □ $p\nmid b_l$ □□□ $p\nmid a_k b_l$), □□. □□ $h(x)$□□□□□. x

□□ **1** □□□□□□ 0□□□□□□ $g(x)$□ Q□□□□□□ $g(x)$□□□□□□□□□□ $g(x)$□□□□□□□□□□□□.

□□ □□□□□□□. □□□□□□.

□□□□□□ $g(x)$□ Q□□□, □□□ $g_i(x)$□ $Q[x]$, $i=1,2$, □□

$$g(x) = g_1(x)g_2(x), \quad \deg g_i(x) < \deg g(x),\ i=1,2.$$

□ $g_i(x) = r_i h_i(x)$, □□ r_i□ Q, $h_i(x)$□□□□□□, $i=1,2$, □

$$g(x) = r_1 r_2 h_1(x)h_2(x).$$

□□ $h_1(x)h_2(x)$□□□□□□□, □□ $r_1r_2 = \pm 1$. □□ $g(x) = [\pm h_1(x)]h_2(x)$. □□ $\deg(\pm h_1(x)) = \deg g_1(x) < \deg g(x)$, $\deg h_2(x) = \deg g_2(x) < \deg g(x)$, □□ $g(x)$□□□□□□□□□□□□□□□□□□. x

□□ **2** □□□□□□□□ 0□□□□□□□ $f(x)$□ Q□□□, □□□□□□□□□□□□□□□□□□□□□.

□□ □ $f(x) = rg(x)$, □□ $g(x)$□□□□□□, r□ Z, □□□ 1□, $g(x) = h_1(x)h_2(x)$, □□ $h_i(x)$□□□□□□, □ $\deg h_i(x) < \deg g(x)$, $i=1,2$. □□ $f(x) = [rh_1(x)]h_2(x)$. □□□ $f(x)$□□□□□□□□□□□□□□□□□□□. x

□□ **3** □□□□□□□ 0□□□□□□ $g(x)$□□□□□□□□ Q□□□□□□□□□□□□□. □□□□□, □□ $g(x)$□□□□□□□□□:

$$g(x) = p_1(x)p_2(x)\ldots p_s(x), \quad g(x) = q_1(x)q_2(x)\ldots q_t(x),$$

□ $s=t$, □□□□□□□□□□□, □

$$p_i(x) = \pm q_i(x), \quad i=1,2,\ldots,s.$$

□□ □□□□.□□□□□□□□□ n□□□□□□. $n=1$□, $g(x)=g(x)$,□ $g(x)$□□□.□□□□□□□□ n□□□□□□□□□□□□.□□□□ n□□□□□□ $g(x)$.□□ $g(x)$□ Q□□□□,□ $g(x)=g(x)$.□□ $g(x)$□ Q□□□,□□□□ 1□, $g(x)=g_1(x)g_2(x)$,□□ $g(x)$□□□□□□,□ $\deg g(x)<\deg g(x)$, $i=1,2$.□ $g(x)$□□□□□□□ $g(x)$□□□□□.

□□□.□ $Q[x]$□□□□□□□□□□, $s=t$,□□□□□□□□□□□□□,□ $p(x)$□ $q(x)$, $i=1,2,\ldots,s$.□□ $p(x)$, $q(x)$□□□□□□□,□□ $p(x)=\pm q(x)$, $i=1,2,\ldots,s$. x

□□□□ 3□□□□□□□□□□□□□□□□□□□□:

□□ **4** □

$$f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$$

□□□□□ n□□ 0□□□□□□□,□□ $\frac{q}{p}$□ $f(x)$□□□□□□,□□ p, q □□□□□□,□□

$$p\mid a_n,\quad q\mid a_0.$$

□□ □ $f(x)=rf_1(x)$,□□ r□ Z^*, $f_1(x)$□□□□□□,□ $\frac{q}{p}$□ $f(x)$□□□□,□ $\frac{q}{p}$□□ $f_1(x)$□□□□.□□□ $Q[x]$□, $\left(x-\frac{q}{p}\right)\Big| f_1(x)$,□□ $(px-q)\mid f_1(x)$.□□ $(p,q)=1$,□□ $px-q$□□□□□□.□□□ 3□□□□□□

$$f_1(x)=(px-q)g(x),$$

□□ $g(x)=b_{n-1}x^{n-1}+\ldots+b_1x+b_0$ □□□□□□.□□

$$f(x)=r(px-q)g(x).\tag{4}$$

□□□□(4)□□□□□□□□□□□□□□□□

$$a_n=rpb_{n-1},\quad a_0=-rqb_0.$$

□□

$$p\mid a_n,\quad q\mid a_0.$$ x

□□ 4□□□□□□□,□□ $\frac{q}{p}$□ $f(x)$□□□□□□,□ $(p,q)=1$,□□□□□□□□□□□ $g(x)$□□

$$f(x)=(px-q)g(x).\tag{5}$$

□(5)□□

$$f(1)=(p-q)g(1),\quad f(-1)=-(p+q)g(-1).$$

□ $\frac{q}{p}$□ ±1□,□

$$\frac{f(1)}{p-q}=g(1)\in Z,\qquad \frac{f(-1)}{p+q}=-g(-1)\in Z.$$

□□□□□□□ $\frac{f(1)}{p-q}\notin Z$ □ $\frac{f(-1)}{p+q}\notin Z$, □□□□ $\frac{q}{p}$□□ $f(x)$□□. □□□□□□□□□□□□□□□□□□□□□□□□.

□ **1** □□ $f(x)=x^3+x+1$□□□□□□□□□□□?

□ □□ $f(x)$□ Q□□□, □ $f(x)$□□□□□□□□□□□□□□□□□□□□□□. □□ $\deg f(x)=3$, □□ $f(x)$□ $Q[x]$□□□□□□□, □□ $f(x)$□□□□□. □□ $a_3=1$, $a_0=1$, □□□□□ 4□, $f(x)$□□□□□□□ ± 1. □□

$$f(1)=3,\qquad f(-1)=-1.$$

□□ ± 1□□□ $f(x)$□□. □□□□□□ $f(x)$□ Q□□□□.

□□: □□□□□□□□ $f(x)$□□□□ 3, □□□□□ $f(x)$□□□□□□□□ $f(x)$□ Q□□□□□□□. □□□□ $f(x)$□□□□□, □□□□ $f(x)$□□□□□□. □□ $f(x)$□□□□□□□ 1□□□, □□ $f(x)$□□□□.

□ **2** □ $f(x)=3x^4+8x^3+6x^2+3x-2$□□□□□□.

□ $a_4=3$□□□□□ $\pm 1,\pm 3$; $a_0=-2$□□□□□ $\pm 1,\pm 2$. □□□ 4□, $f(x)$□□□□□□□□:

$$\pm 1,\qquad \pm 2,\qquad \pm\frac{1}{3},\qquad \pm\frac{2}{3}.$$

□□ $f(1)=18\neq 0$, $f(-1)=-4\neq 0$, □□ ± 1□□ $f(x)$□□.

□□ 2, □□

$$\frac{f(-1)}{p+q}=-\frac{4}{3}$$

□□□□, □□ 2□□ $f(x)$□□.

□□ -2, □□

$$\frac{f(1)}{p-q}=\frac{18}{3}=6,\qquad \frac{f(-1)}{p+q}=\frac{-4}{-1}=4,$$

□□□□□□□□□□□□□□ -2□□□ $f(x)$□□:

$$\begin{array}{rrrrr|r}
3 & 8 & 6 & 3 & -2 & -2\\
 & -6 & -4 & -4 & 2 & \\
\hline
3 & 2 & 2 & -1 & 0 & \\
 & -6 & 8 & -20 & & \\
\hline
3 & -4 & 10 & -21 & &
\end{array}$$

□□□ -2□ $f(x)$□□□. □□

$$f(x)=(x+2)(3x^3+2x^2+2x-1).$$

□□ $\frac{1}{3}$, □□

$$\frac{f(1)}{p-q}=9,\quad \frac{f(-1)}{p+q}=-1,$$

□□□□□□□□:

$$\begin{array}{rrrr|l} 3 & 2 & 2 & -1 & \frac{1}{3} \\ & 1 & 1 & 1 & \\ \hline 3 & 3 & 3 & \multicolumn{1}{|r}{0} & \\ & 1 & \frac{4}{3} & & \\ \cline{1-3} 3 & 4 & \frac{13}{3} & & \end{array}$$

□□□ $\frac{1}{3}$ □ $f(x)$ □□□. □□

$$f(x)=(x+2)\left(x-\frac{1}{3}\right)(3x^2+3x+3).$$

□□ x^2+x+1 □□□□□. □□ $f(x)$ □□□□□□□ $-2,\ \frac{1}{3}$, □□□□□□.

□□□□□□□□□□□□□□□□□□□□□□□□□□□□.

□□ 5(Eisenstein □□□) □

$$f(x)=a_nx^n+\dots+a_1x+a_0$$

□□□□□ n □□□□□□□□□□, □□□□□□□□ p, □□

1) $p\nmid a_n$;

2) $p\mid a_i,\ i=0,1,\dots,n-1$;

3) $p^2\nmid a_0$;

□ $f(x)$ □□□□□□□□□□□.

□□ □□ $f(x)$ □ Q □□□, □□□□ 2 □

$$f(x)=(b_mx^m+\dots+b_1x+b_0)(c_lx^l+\dots+c_1x+c_0),\tag{6}$$

□□ b_i, c_j □□□□ $(i=0,1,\dots,m;\ j=0,1,\dots,l)$, b_m □ 0, c_l □ 0, $m<n$, $l<n$, □□ $m+l=n$. □□

$$a_n=b_mc_l,\quad a_0=b_0c_0.$$

□□ $p\mid a_0$, □□ $p\mid b_0$ □□ $p\mid c_0$. □□□ $p^2\nmid a_0$, □□ p □□□□□□ b_0 □ c_0. □□□ $p\mid b_0$ □ $p\nmid c_0$. □□ $p\nmid a_n$, □□ $p\nmid b_m$. □□ $b_0, b_1, \dots, b_m$ □□□□□□

□ p□□□□ b_k,□

$$p \mid b_0,\ p \mid b_1,\ \dots,\ p \mid b_{k-1},\ p \nmid b_k,\ 0 < k \text{□} m \tag{7}$$

□□(6)□□□ x^k □□□,□

$$a_k = b_k c_0 + b_{k-1} c_1 + \dots + b_0 c_k, \tag{8}$$

□□ k□ $m < n$,□□ $p \mid a_k$.□□□(7)□(8)□□ $p \mid b_k c_0$.□□ $p \nmid b_k$,□□ $p \mid c_0$,□□.□□ $f(x)$□ Q□□□□. x

□□□□ 5□□□□□

□□ **6** □ $Q[x]$□□□□□□□□□□□□□□.

□□ □□□□□□□ n,□ $f(x) = x^n + 2$.□□ 2□□□□ 5□□□,□□ $x^n + 2$□ Q□□□□. x

□□□□□ Eisenstein□□□□□□□ $f(x)$□□□ Q□□□□,□□□□□□□□□ x□ $Q[x]$□□□□ $x+b$□□,□□□□□□□□

$$g(x) := f(x+b)$$

□ $g(x)$□□□ Eisenstein□□□□□□□□□□,□□□□□ $f(x)$□□□□□?□□□□□□□□□□□□□:

□□ **7** □ $f(x) = a_n x^n + \dots + a_1 x + a_0$ □□□ n□□□□□□□□□□,□ b□□□□□□□□□□□.□

$$\begin{aligned} g(x) &:= f(x+b) \\ &= a_n (x+b)^n + \dots + a_1 (x+b) + a_0, \end{aligned}$$

□ $f(x)$□ Q□□□□□□□□□□□□□ $g(x)$□ Q□□□□.

□□ □□□□□.□□ $g(x)$□□□□ $a_n x^n$,□□

$$\deg g(x) = n = \deg f(x).$$

□□ $f(x)$□ Q□□□,□

$$f(x) = f_1(x) f_2(x), \quad \deg f_i < \deg f, \quad i = 1, 2. \tag{9}$$

x□ $x+b$□□,□(9)□□

$$f(x+b) = f_1(x+b) f_2(x+b), \tag{10}$$

(10)□□□ $g(x)$□ Q□□□.

□□□□□.x□ $x-b$□□,□□

$$g(x) = f(x+b),$$

□

$$g(x-b) = f(x).$$

□□□□□□□□□,□□ $f(x)$□ Q□□□,□ $g(x)$□ Q□□□□□. x

□ **3** □ p□□□□□,□□□

$$f(x) = x^{p-1} + x^{p-2} + \dots + x + 1$$

□□□□□□□□□.□□ $f(x)$□ Q□□□□.

□□ □□□

$$(x-1)f(x)=(x-1)(x^{p-1}+x^{p-2}+\dots+x+1)$$
$$=x^p-1. \tag{11}$$

x□ x+1□□,□(11)□□

$$xf(x+1)=(x+1)^p-1. \tag{12}$$

□(12)□□

$$f(x+1)=x^{p-1}+px^{p-2}+\dots+C_p^k x^{p-k-1}+\dots+p.$$

□ $g(x):=f(x+1)$

$$=x^{p-1}+px^{p-2}+\dots+C_p^k x^{p-k-1}+\dots+p,$$

□□ $C_p^k=\dfrac{p(p-1)\dots(p-k+1)}{k!}$, 1□ $k<p$,

□□ $(p,k!)=1$, □□

$$k!\mid(p-1)\dots(p-k+1),$$

□□ $p\mid C_p^k$, 1□ $k<p$.

□□□□ $g(x)$,□□ p□□ Eisenstein□□□□□,□□ $g(x)$□ Q□□□□.□□ $f(x)$□ Q□□□□. x

□□□□□□□□□□,□□□□□7□□□□□ Eisenstein□□□□□□□□□□□.□□□□□□□□□□□□□ $f(x)$,□□□□□,□□□□□□□□□,□□□□□□□□.□□□□□□ $f(x)$□□□□□□,□□□□□□□□□□□□□□□□□□□□□□□□□□.□□□□ 3□□□□□□□□,□□□□□□□□□□□□ 1□□□□.

□ □ 7.8

1. □□□□□□□□□□□:

(1) $2x^3+x^2-3x+1$;

(2) $2x^4-x^3-19x^2+9x+9$.

2. □□□□□□□□□□□□□□□□□?

(1) $x^4-6x^3+2x^2+10$; (2) $7x^5+18x^4+6x-6$;

(3) x^5+5x^3+1; (4) x^4-2x^3+2x-3;

(5) x^p+px^2+1, p□□□□;

(6) x^3+x^2-3x+2;

(7) $2x^3-x^2+x+1$; (8) x^4-5x+1.

3. □ $n>1$,□□:n□□□□□□□□□□□□□□□□□□.

4. □ $f(x)=a_nx^n+\dots+a_1x+a_0$□□□□□□□□□□□□□□□,□□:□□

$a_n+a_{n-1}+\dots+a_1+a_0$ □□□□□,□ 1□ -1□□□ $f(x)$□□.

5. □ $f(x)$□□□□□□□□□□□□□□ 1□□□□□□,□□:□□ $f(0)$□ $f(1)$□□□□,□ $f(x)$□□□□□.

6. □ $f(x)=x^3+ax^2+bx+c$□□□□□□□,□□:□□$(a+b)c$□□□,□ $f(x)$□□□□□□□□□.

7. □ $a_1,a_2,\dots,a_n$□ n□□□□□□,□

$$f(x)=(x-a_1)(x-a_2)\dots(x-a_n)+1,$$

□□:□□ n□□□,□ $f(x)$□□□□□□□□□.□□ n□□□, $f(x)$□□□□□□□□□□□?

*8. □ $a_1,a_2,\dots,a_n$□ n□□□□□□,□

$$f(x)=(x-a_1)(x-a_2)\dots(x-a_n)-1,$$

□□ $f(x)$□□□□□□□□□.

*9. □ $a_1,a_2,\dots,a_n$□ n□□□□□,□

$$f(x)=\prod_{i=1}^{n}(x-a_i)^2+1,$$

□□ $f(x)$□ Q□□□□.

§9 □□□□□

□□□□□□□□□ 6□□,□□□□□□ K□□ n□□□□

$$f(x_1,x_2,\dots,x_n)=\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}x_ix_j,$$

□□ a_{ij}□ K, 1□ i,j□ n.□□□□□□□□.

□□ **1** □ K□□□□□,□□□□ K□ n□□□ $x_1,x_2,\dots,x_n$□□□□

$$\sum_{i_1,i_2,\dots,i_n}a_{i_1i_2\cdots i_n}x_1^{i_1}x_2^{i_2}\dots x_n^{i_n},\qquad(1)$$

□□ $a_{i_1i_2\cdots i_n}$□ K, $i_1,i_2,\dots,i_n$□□□□□,(1)□□□□□□□□□□□□□,(1)□□□□□ **K**□□ **n**□□□□,□□□□□□□□□:□□□□□□□□□□□□□ 0,□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□,□□□□ 0□□□□□□□□□□□□□□.□□,□□ $x_1,x_2,\dots,x_n$□□ **n**□□□□□□.

□□□ K□□ n□□□□□,□□□□□□□□ $x_j(j=1,2,\dots,n)$□□□□□□□□□,□□□□□□□□□□□□.□□□□ n□□□□□□□□□□□□□□(□□□□□□□□□□□□).

□□ K□□□ n□□□□□□□□□□□□□□ 0,□□□□□□□□,□□ 0.

□□□□ $f(x_1, x_2, \dots, x_n)$, $g(x_1, x_2, \dots, x_n)$, …□□ f, g, …□□□ n□□□□.

□□□□□ $a_{i_1 i_2 \cdots i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$, $i_1 + i_2 + \dots + i_n$ □□□□□□□(□□□□□□□).

□□ n□□□□ $f(x_1, x_2, \dots, x_n)$□□□□□ 0□□□□□□□□□□□□□ $f(x_1, x_2, \dots, x_n)$□□□□(□□□□□□□),□ $\deg f$ □□□.

□□□□□□□□(□□□□□)□□□ -□.

□□,3□□□□

$$5x_1^4 + 3x_1^3 x_2 + 2x_1 x_2 x_3^2 + x_2^3 + x_2 x_3$$

□□□□ 4,□□□□□ $5x_1^4$, $3x_1^3 x_2$, $2x_1 x_2 x_3^2$ □□□□□ 4.

□□ K□□□ n□□□□□□□□□□□ $K[x_1, x_2, \dots, x_n]$. □ $K[x_1, x_2, \dots, x_n]$□□□□□□□□□□:

$$\sum_{i_1, i_2, \dots, i_n} a_{i_1 i_2 \cdots i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} + \sum_{i_1, i_2, \dots, i_n} b_{i_1 i_2 \cdots i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$$

$$\overset{\mathrm{def}}{=} \sum_{i_1, i_2, \dots, i_n} (a_{i_1 i_2 \cdots i_n} + b_{i_1 i_2 \cdots i_n}) x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \tag{2}$$

$$\sum_{i_1, i_2, \dots, i_n} a_{i_1 i_2 \cdots i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \quad \sum_{j_1, j_2, \dots, j_n} b_{j_1 j_2 \cdots j_n} x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}$$

$$\overset{\mathrm{def}}{=} \sum_{s_1, s_2, \dots, s_n} c_{s_1 s_2 \cdots s_n} x_1^{s_1} x_2^{s_2} \cdots x_n^{s_n}, \tag{3}$$

□□ $$c_{s_1 s_2 \cdots s_n} = \sum_{i_1 + j_1 = s_1} \sum_{i_2 + j_2 = s_2} \cdots \sum_{i_n + j_n = s_n} a_{i_1 i_2 \cdots i_n} b_{j_1 j_2 \cdots j_n}. \tag{4}$$

□□□□ $K[x_1, x_2, \dots, x_n]$□□□□□□□□□□□□□□□□□.□□□□□□□□□□.□□□□□□,□□□□□□ 1.□□□□□,□□□□□□□ K□□ n□□□□□.

□□□

$$\deg(f + g) \,\square\, \max\{\deg f, \deg g\},$$

□□□□□□□ $\deg fg$ □ $\deg f$, $\deg g$ □□□.□□□ n□□□□ $f(x_1, x_2, \dots, x_n)$□□□□□□□□□□□□,□□□□□□□□.

□□□□□□ $a x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ □□□□ n□□□□□□□ $(i_1, i_2, \dots, i_n)$,□□□□□□□□□.□□□□□□□□□□□□□□□□□□□□,□□□□□ n□□□□□□□□□□□□□□□□□□□.

□□□□ n□□□□□□□□ $(i_1, i_2, \dots, i_n)$□ $(j_1, j_2, \dots, j_n)$,□□ $i_1 =$

$j_1, i_2=j_2, \ldots, i_{s-1}=j_{s-1}, i_s>j_s$ (1□ s□ n), □□ $(i_1, i_2, \ldots, i_n)$ □□ $(j_1, j_2, \ldots, j_n)$, □□

$$(i_1, i_2, \ldots, i_n)>(j_1, j_2, \ldots, j_n).$$

□□□□□□□□□, □□□□□□ n □□□□□□□□ $(i_1, i_2, \ldots, i_n)$, $(j_1, j_2, \ldots, j_n)$, □□

$$(i_1, i_2, \ldots, i_n)>(j_1, j_2, \ldots, j_n),$$

$$(i_1, i_2, \ldots, i_n)=(j_1, j_2, \ldots, j_n),$$

$$(j_1, j_2, \ldots, j_n)>(i_1, i_2, \ldots, i_n)$$

□□□□□□□□□□□.

□□“ >”□□□□□, □□□

$$(i_1, i_2, \ldots, i_n)>(j_1, j_2, \ldots, j_n),$$

□□ $$(j_1, j_2, \ldots, j_n)>(k_1, k_2, \ldots, k_n),$$

□ $$(i_1, i_2, \ldots, i_n)>(k_1, k_2, \ldots, k_n).$$

□□□□ $i_l-k_l=(i_l-j_l)+(j_l-k_l)$, □□□□□□. □□,

$$(4, 2, 3, 3)>(4, 2, 2, 4), \quad (4, 2, 2, 4)>(4, 1, 4, 3),$$

□□□□□ $(4, 2, 3, 3)>(4, 1, 4, 3)$. □□□□□□□□□ $(4, 2, 3, 3)>(4, 1, 4, 3)$.

□□□□□□□□□ n □□□□□□□□□□□□□□□. □□□, n □ □□□□□□□□□□□□□□□□. □□□□□□□□□□□□□□□□□ □□□□□□□□, □□□□□□□□□□.

□□, □□□

$$2x_1^4x_2x_3+x_1x_2^5x_3+6x_1^3$$

□□□□□□□□□□□

$$2x_1^4x_2x_3+6x_1^3+x_1x_2^5x_3.$$

□□□□□□□□□□□□□□□□□□□□□□□□ n □□□□□□ □. □□, □□□□□□□□□□ $2x_1^4x_2x_3$. □□□, □□□□□□□□□□□ □. □□, □□□□□□□□□ 7, □□□□□□□ 6.

□□ **1** □ $K[x_1, x_2, \ldots, x_n]$ □□□□□□□□□□□□□□□□□□□ □□□□□. □□□□□□□□□□□□□□□□□□□, □ $K[x_1, x_2, \ldots, x_n]$ □□□□□□.

□□ □ $f(x_1, x_2, \ldots, x_n)$, $g(x_1, x_2, \ldots, x_n)$ □ $K[x_1, x_2, \ldots, x_n]$ □□□□ □□□□. □ $f(x_1, x_2, \ldots, x_n)$ □□□□ $ax_1^{p_1}x_2^{p_2}\ldots x_n^{p_n}$, a□ 0. □ $g(x_1, x_2, \ldots, x_n)$ □□□□ $bx_1^{q_1}x_2^{q_2}\ldots x_n^{q_n}$, b□ 0. □□□□ fg □□□□ $abx_1^{p_1+q_1}x_2^{p_2+q_2}\ldots x_n^{p_n+q_n}$,

□□□□

$$(p_1+q_1,\ p_2+q_2,\ \dots,\ p_n+q_n).$$

□□ fg□□□□□□□□□□□□□. fg□□□□□□□□□□□□□□

□□□□□.

$$(p_1+j_1,\ p_2+j_2,\ \dots,\ p_n+j_n),$$

□□
$$(i_1+q_1,\ i_2+q_2,\ \dots,\ i_n+q_n),$$

□□
$$(i_1+j_1,\ i_2+j_2,\ \dots,\ i_n+j_n),$$

□□
$$(p_1,\ p_2,\ \dots,\ p_n)>(i_1,\ i_2,\ \dots,\ i_n),$$
$$(q_1,\ q_2,\ \dots,\ q_n)>(j_1,\ j_2,\ \dots,\ j_n).$$

□□□

$$(p_1+q_1,\ p_2+q_2,\ \dots,\ p_n+q_n)>(p_1+j_1,\ p_2+j_2,\ \dots,\ p_n+j_n),$$
$$(p_1+q_1,\ p_2+q_2,\ \dots,\ p_n+q_n)>(i_1+q_1,\ i_2+q_2,\ \dots,\ i_n+q_n),$$
$$(i_1+q_1,\ i_2+q_2,\ \dots,\ i_n+q_n)>(i_1+j_1,\ i_2+j_2,\ \dots,\ i_n+j_n).$$

□□□□□

$$(p_1+q_1,\ p_2+q_2,\ \dots,\ p_n+q_n)>(i_1+j_1,\ i_2+j_2,\ \dots,\ i_n+j_n).$$

□□□□□ $abx_1^{p_1+q_1}x_2^{p_2+q_2}\cdots x_n^{p_n+q_n}$ □□□□ fg□□□□□□□□, □□□ □□ fg□□□□□, □□□□ fg□□□. ×

□□ $K[x_1,\ x_2,\ \dots,\ x_n]$□□□□□, □□□□□□□.

□□□□□□□□□

□□ **2** □ $K[x_1,\ x_2,\ \dots,\ x_n]$□, □□ f_i□0, $i=1,2,\dots,m$, □ $f_1f_2\cdots f_m$ □□□□□□□ f_i □□□□□□. ×

□□□□□□□□□□□□□□□□:

□□ **2** □□ K□□ n□□□□ $g(x_1,\ x_2,\ \dots,\ x_n)$□□ m□□□□□□, □□□□□□□□□□ 0□□□□□□ m□□.

□□, $g(x_1,\ x_2,\ x_3)=2x_1^4+3x_1^2x_2x_3+x_1x_2x_3^2$ □□□ 4□□□□□□.

□□, $K[x_1,\ x_2,\ \dots,\ x_n]$□□□□□□□□□□□□□□□□□□, □□□ □□□□□□□□□□□□□□.

□□□□□□ n□□□□ $f(x_1,\ x_2,\ \dots,\ x_n)$, □□□ f□□□□□□□□ □□□□□□□, □ f□□□□□□□□

$$f(x_1,\ x_2,\ \dots,\ x_n)=\sum_{i=0}^{m}f_i(x_1,\ x_2,\ \dots,\ x_n),\qquad m=\deg f,\tag{5}$$

□□ $f_i(x_1,\ x_2,\ \dots,\ x_n)$□ i□□□□□□, □□□ $f(x_1,\ x_2,\ \dots,\ x_n)$□ i□□□ □□.

□□(5)□□□□□□□□□□□.

3

$$f(x_1, x_2, \dots, x_n), \quad g(x_1, x_2, \dots, x_n) \in K[x_1, x_2, \dots, x_n],$$

$$\deg fg = \deg f + \deg g. \tag{6}$$

f, g , (6) .

$$f \neq 0, \quad g \neq 0, \quad \deg f = m, \quad \deg g = s,$$

$$f = f_0 + f_1 + \dots + f_m, \quad g = g_0 + g_1 + \dots + g_s,$$

f_i f i , g_j g j , $f_m \neq 0$, $g_s \neq 0$.

$$fg = f_0 g_0 + \dots + f_0 g_s + f_1 g_0 + \dots + f_1 g_s$$

$$+ \dots + f_m g_0 + \dots + f_m g_s, \tag{7}$$

$f_i g_j$ $i + j$ $(i = 0, 1, \dots, m; j = 0, 1, \dots, s)$. $f_m \neq 0$, $g_s \neq 0$, $f_m g_s \neq 0$. $f_m g_s$ $m + s$.

$$\deg fg = m + s = \deg f + \deg g.$$

* $n > 1$, $K[x_1, x_2, \dots, x_n]$, .

.

§1 , K $K[x]$. , K n $K[x_1, x_2, \dots, x_n]$, .

4 K , R , R K (R R_1 K , R_1 K , R R_1). $t_1, \dots, t_n$ R ,

$$\sigma_{t_1, \dots, t_n}: K[x_1, x_2, \dots, x_n] \longrightarrow R$$

$$f(x_1, x_2, \dots, x_n) \longmapsto f(t_1, t_2, \dots, t_n), \tag{8}$$

$$f(x_1, x_2, \dots, x_n) = \sum_{i_1, i_2, \dots, i_n} a_{i_1 i_2 \dots i_n} x_1^{i_1} x_2^{i_2} \dots x_n^{i_n},$$

$$f(t_1, t_2, \dots, t_n) \overset{\text{def}}{=} \sum_{i_1, i_2, \dots, i_n} a_{i_1 i_2 \dots i_n} t_1^{i_1} t_2^{i_2} \dots t_n^{i_n},$$

$\sigma_{t_1, \dots, t_n}$ $K[x_1, x_2, \dots, x_n]$ R ,

$$\sigma_{t_1, \dots, t_n}(x_i) = t_i, \quad i = 1, \dots, n.$$

,

$$f(x_1, \dots, x_n) + g(x_1, \dots, x_n) = h(x_1, \dots, x_n),$$

$$f(x_1, \dots, x_n) g(x_1, \dots, x_n) = p(x_1, \dots, x_n),$$

□□ $$f(t_1,\dots,t_n)+g(t_1,\dots,t_n)=h(t_1,\dots,t_n),$$
$$f(t_1,\dots,t_n)g(t_1,\dots,t_n)=p(t_1,\dots,t_n).$$

□(8)□□□□□ $\sigma_{t_1,\dots,t_n}$ □□ $x_1,\dots,x_n$ □ $t_1,\dots,t_n$ □□.

□□ □□□§1□□3□□□□□. x

$K[x_1,x_2,\dots,x_n]$□□□□□□□□□□□□□□□□□□□□ $K[x_1,x_2,\dots,x_n]$□□□□□,□□ K□□□□□,□□ $K[x_1,x_2,\dots,x_n]$□□□□□ K□□□□□.□□ $x_1,x_2,\dots,x_n$ □□□ $K[x_1,x_2,\dots,x_n]$□□□ n□□□□□,□□□□□□□□□□□□□□□□.

□□□□□□□□□□: $x_1,x_2,\dots,x_n$ □ K□□□ n□□□ $c_1,c_2,\dots,c_n$ □□,□□□□□□□□□□□□□□□□□□:

□ $f(x_1,x_2,\dots,x_n)$□□□ K□□□□ n□□□□,□□ K□□□ n□□□ $c_1,c_2,\dots,c_n$,□ $x_1,x_2,\dots,x_n$ □ $c_1,c_2,\dots,c_n$ □□,□ $f(c_1,c_2,\dots,c_n)$□ K. □□ n□□□□ $f(x_1,x_2,\dots,x_n)$□□□□□ K^n □□□ K□□□□□(□ K □□ n□□□),□□ f□□,□

$$f:\ K^n \qquad\qquad K$$
$$(c_1,c_2,\dots,c_n)\longmapsto f(c_1,c_2,\dots,c_n). \tag{9}$$

□□□□□ K□□ n□□□□□□□□ K□□ n□□□□□□□□ $\mathbf{K}$□□ $\mathbf{n}$□□□□□□.

□ $f(x_1,x_2,\dots,x_n)$, $g(x_1,x_2,\dots,x_n)$□□□ K□□□□ n□□□□.□□,□□ $f(x_1,x_2,\dots,x_n)$□ $g(x_1,x_2,\dots,x_n)$□□,□□□□□□ n□□□□□□ f□ g□□□,□□□□?□□□□□□□□□□:

□□ **1** □ $h(x_1,\dots,x_n)$□□□ K□□□□ n□□□□,□□ $h(x_1,\dots,x_n)$□0,□ h□□□□□.

□□ □□□□□□□ n□□□□. $n=1$□,□□□□ K□□□□□□□□ $h(x)$□□□□□□□□□□,□□□□ c□ K,□ $h(c)$□0.

□□□□□ $K[x_1,\dots,x_{n-1}]$□□□□□□□,□□□ $K[x_1,\dots,x_{n-1},x_n]$□□□□□ $h(x_1,\dots,x_n)$.□ $h(x_1,\dots,x_n)$□□

$$h(x_1,\dots,x_n)=u_0(x_1,\dots,x_{n-1})+u_1(x_1,\dots,x_{n-1})x_n$$
$$+\dots+u_s(x_1,\dots,x_{n-1})x_n^s,$$

□□ $u_i(x_1,\dots,x_{n-1})$□ $K[x_1,\dots,x_{n-1}]$, $i=0,1,\dots,s$,□ $u_s(x_1,\dots,x_{n-1})$□0.□□□□□,u_s□□□□□,□□□□ $c_1,\dots,c_{n-1}$□ K,□□ $u_s(c_1,\dots,c_{n-1})$□0.□□ $K[x_n]$□□□□□

$$h(c_1,\dots,c_{n-1},x_n)=u_0(c_1,\dots,c_{n-1})+u_1(c_1,\dots,c_{n-1})x_n$$

$$+\dots+u_s(c_1,\dots,c_{n-1})x_n^s$$

□□□□□,□□□□ c_n□ K,□□

$$h(c_1,\dots,c_{n-1},c_n)=u_0(c_1,\dots,c_{n-1})+u_1(c_1,\dots,c_{n-1})c_n$$
$$+\dots+u_s(c_1,\dots,c_{n-1})c_n^s \ □\ 0.$$ x

□□:□□ 1□□□□□□□ K□□□□□□□.

□□□ 1□□□□

□□ **5** □ $f(x_1,\dots,x_n)$, $g(x_1,\dots,x_n)$□ $K[x_1,\dots,x_n]$,□□□□□ $f(x_1,\dots,x_n)$□ $g(x_1,\dots,x_n)$□□□,□□□□□□□ n□□□□□□ f□ g□□□□.

□□ □□□□□

$$h(x_1,\dots,x_n)=f(x_1,\dots,x_n)-g(x_1,\dots,x_n),$$

□□□□□ f□ g□□□,□ $h(x_1,\dots,x_n)$□0.□□□ 1, h□□□□□,□□□□ $c_1,\dots,c_n$□ K,□□ $h(c_1,\dots,c_n)$□0. $x_1,\dots,x_n$□ $c_1,\dots,c_n$□□,□□□□□□□□

$$f(c_1,\dots,c_n)\ □\ g(c_1,\dots,c_n),$$

□□□□ f□ g□□□. x

□□□□□ K□□□ n□□□□□□□□□□□□□□ K_{npol},□□□□□□□□□□□□□□□□□: " $(c_1,\dots,c_n)$□ K^n,

$$(f+g)(c_1,\dots,c_n)\ \overset{\text{def}}{□}\ f(c_1,\dots,c_n)+g(c_1,\dots,c_n) \tag{10}$$

$$(fg)(c_1,\dots,c_n)\ \overset{\text{def}}{□}\ f(c_1,\dots,c_n)g(c_1,\dots,c_n) \tag{11}$$

□□□□ K_{npol}□□□□,□□□□□ **K**□□ **n**□□□□□□□.□□□§6□□□ 7□□□□□□□:□□ K□□ n□□□□□ $K[x_1,\dots,x_n]$□ K□□ n□□□□□□□ K_{npol}□□□□.□□□□□□□□□ K□□ n□□□□□ K□□ n□□□□□□□□□□.

□ $f(x_1,\dots,x_n)$□ $K[x_1,\dots,x_n]$,□□ $c_1,\dots,c_n$□ K,□□ $f(c_1,\dots,c_n)=0$,□□ $(c_1,\dots,c_n)$□ $f(x_1,\dots,x_n)$□□□□□.□ K□□□□,□ $n=2$,□ $f(x,y)$□□□□□□□□□□□□□□□□□□□□;□ $n=3$,□ $f(x,y,z)$□□□□□□□□□□□□□□□□□□□□.□□□□ K□□□ n□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□.

□□□□□ $x_1,\dots,x_n$□ $K[x_1,\dots,x_n]$□ n□□□□□□□□□□□□,□□□□□□□□□□□□□□□□□□□:

□□ **6** □ $f(x_1,\dots,x_n)$□ $K[x_1,\dots,x_n]$□ f□0, m□□□□□□□.□ $f(x_1,\dots,x_n)$□ m□□□□□□□□□□□□□□□□□□□□□ t□ K,□

$$f(tx_1,tx_2,\ldots,tx_n)=t^m f(x_1,x_2,\ldots,x_n). \tag{12}$$

*□□ □□□.□ $f(x_1,\ldots,x_n)$□ m□□□□□,□

$$f(x_1,\ldots,x_n)=\sum_{i_1,i_2,\ldots,i_n} a_{i_1i_2\cdots i_n}x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n},\quad i_1+i_2+\cdots+i_n=m,$$

□□ $t\in K$,□□□ $x_1,x_2,\ldots,x_n$ □ $tx_1,tx_2,\ldots,tx_n$ □□,□□□□

$$f(tx_1,\ldots,tx_n)=\sum_{i_1,i_2,\ldots,i_n} a_{i_1i_2\cdots i_n}(tx_1)^{i_1}\cdots(tx_n)^{i_n}$$
$$=t^m f(x_1,\ldots,x_n).$$

□□□.□□□□ $t\in K$,□(12)□□□.□ $f(x_1,\ldots,x_n)$□□

$$f=f_0+f_1+\cdots+f_s, \tag{13}$$

□□ f_i □ f □ i □□□□□,$i=0,1,\ldots,s$.□□ $t\in K$, $x_1,x_2,\ldots,x_n$ □ $tx_1,tx_2,\ldots,tx_n$ □□,□(13)□□□

$$f(tx_1,\ldots,tx_n)=f_0(tx_1,\ldots,tx_n)+f_1(tx_1,\ldots,tx_n)$$
$$+\cdots+f_s(tx_1,\ldots,tx_n).$$

□□□□□□□□□□□□□□□□

$$t^m f(x_1,\ldots,x_n)=f_0(x_1,\ldots,x_n)+tf_1(x_1,\ldots,x_n)$$
$$+\cdots+t^s f_s(x_1,\ldots,x_n). \tag{14}$$

□(13)□□□(14)□□□,□□□□ $K[x_1,\ldots,x_n]$□□□□□□□□□□□,□□□□

$$t^m f_i(x_1,\ldots,x_n)=t^i f_i(x_1,\ldots,x_n),\quad i=0,1,\ldots,s. \tag{15}$$

□□ $i\in\{0,1,\ldots,s\}$□ $i\neq m$ □□ $f_i\neq 0$,□□(15)□□□□□ f_i □ $t^m=t^i$, $\forall t\in K$.□§6□□□6□,$x^m=x^i$,□□ $i\neq m$ □□.□□ $f_i=0$,□ $i\neq m$□□ $f=f_m$.□□ $f\neq 0$,□□ f □ m□□□□□□. ×

□□□□,□ R□(□□□□□)K□□□□□□□,□□ R□ n□□□ $t_1,t_2,\ldots,t_n$,□□ $f(x_1,x_2,\ldots,x_n)\in K[x_1,x_2,\ldots,x_n]$,□ $x_1,x_2,\ldots,x_n$ □ $t_1,t_2,\ldots,t_n$ □□,□□□ $f(t_1,t_2,\ldots,t_n)$□□ $t_1,t_2,\ldots,t_n$ □ K□□□□□□.□□□□□□□,$t_1,t_2,\ldots,t_n$ □□□□ $f(t_1,t_2,\ldots,t_n)$□ n□□□□ $f(x_1,x_2,\ldots,x_n)$□□,□□□□□□□□:$t_1,t_2,\ldots,t_n$ □□□□□□,□□□□□□□□,□□□ n□□□□,□□□□□.

□□ 7.9

1. □□□4□□□□□□□□□□□□□□□□□□□□□:

(1) $f(x_1, x_2, x_3, x_4) = x_3^4 x_4 - x_1^3 x_2 + 5x_2 x_3 x_4 + 2x_2^4 x_3 x_4$;

(2) $f(x_1, x_2, x_3, x_4) = x_1^3 + x_3^2 + 3x_1 x_2^2 x_4 - 5x_1^2 x_3 x_4^2 - 2x_2^3 x_3$.

2. □ 3□□□□□□ $f(x_1, x_2, x_3) = x_1^3 + x_2^3 + x_3^3 - 3x_1 x_2 x_3$ □□□□ 3□□□□□ □□□□.

3. □ f, g□ $K[x_1, \ldots, x_n]$, □ g□0. □□: □□□□□ $g(c_1, \ldots, c_n)$□0□□□□□□ □ $c_1, \ldots, c_n$□ K, □□ $f(c_1, \ldots, c_n) = 0$, □ $f(x_1, \ldots, x_n) = 0$.

*4. □ f, g□ $K[x_1, \ldots, x_n]$, □□□□ h□ $K[x_1, \ldots, x_n]$, □□ $f = gh$, □□ g□□ f, □□ $g | f$. □□□ g□ f□□□□□. □□ $f | g$, □□ $g | f$, □□ f□ g□□□□, □□ f□g.

(1) □□: f□g $\Longleftrightarrow$ f□ g□□□□□□□□□□;

(2) f□ $K[x_1, \ldots, x_n]$□□□□□□, □□ f□□□□□□□□□□□□□□ f□□□□. □ □□□□□□. □□: □ $K[x, y]$□, □□□ $x^2 - y$□□□□□.

§ 10 □□□□□

□□□□□□□□□□□□□□□□□, □□□□□□. □□□□□□□ □□□□□, □□□□□□□□.

□□ **1** □□□□□□ Ω □□□□□□□□□□ Ω □□□□□.

□ Ω □□□ n□□□□□□□. □□□□□□□□□□□□□□□□□ □, □□□□□□□□ $\Omega = \{1, 2, \ldots, n\}$. □ σ □ Ω □□□□□(□□ σ □ n□□ □), □ i □□□σ□□□□ a_i, □

$$\sigma(i) = a_i, \qquad i = 1, 2, \ldots, n.$$

□□□□□□□ σ □□□□:

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}. \tag{1}$$

□□ σ □ Ω □□□□□□□□□, □□ $a_1 a_2 \ldots a_n$ □□□□ $1, 2, \ldots, n$ □□□□□ □, □ n□□□. □□, □□□□□□ n□□□ $a_1 a_2 \ldots a_n$, □ $\sigma: i \longmapsto a_i$ □ Ω □□□□□. □□ n□□□□□□□□□ n□□□□□□□□□□□□□□□ □□. □□□ S_n □□□□ n□□□□□□□, □□□□□□□□□□, $|S_n| = n!$.

□□,

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} \tag{2}$$

□□□ 4□□□, $\sigma(1) = 3, \sigma(2) = 1, \sigma(3) = 4, \sigma(4) = 2$.

□ σ □ Ω □□□□□, □□ σ □ Ω □□□□□□□□, □□ σ □□□□□.

σ □□□□σ^{-1}□□□□σ□□□□.□□,(2)□□□□□σ□□□□σ^{-1}□□:

$$\sigma^{-1}(3)=1,\quad \sigma^{-1}(1)=2,\quad \sigma^{-1}(4)=3,\quad \sigma^{-1}(2)=4$$

□□σ^{-1}□□□□□

$$\sigma^{-1}=\begin{pmatrix}1&2&3&4\\2&4&1&3\end{pmatrix}. \tag{3}$$

□□□□□□(2)□□(3)□□□□$\sigma^{-1}\sigma=1_\Omega$,$\sigma\sigma^{-1}=1_\Omega$,□□$1_\Omega$□□$\Omega$□□□□□□□□.

□□□□n□□□□i□□j,□j□□i,□□□□□□□,□□□□□□□□□□□,□□(ij).□□$(ij)^{-1}=(ij)$.

□□□□□□$\sigma\in S_n$,□□$K[x_1,\ldots,x_n]$□□□□□K□□□□□,□n□□□□□□□□□□,$x_1,x_2,\ldots,x_n$□$x_{\sigma^{-1}(1)},x_{\sigma^{-1}(2)},\ldots,x_{\sigma^{-1}(n)}$□□□$K[x_1,x_2,\ldots,x_n]$□$K[x_1,x_2,\ldots,x_n]$□□□□□,□□□□□□□□□□□□,□□□□□□σ.□

$$\sigma: K[x_1,x_2,\ldots,x_n] \to K[x_1,x_2,\ldots,x_n]$$
$$f(x_1,x_2,\ldots,x_n)\longmapsto f(x_{\sigma^{-1}(1)},x_{\sigma^{-1}(2)},\ldots,x_{\sigma^{-1}(n)}),$$

□□□□

$$(\sigma f)(x_1,x_2,\ldots,x_n)=f(x_{\sigma^{-1}(1)},x_{\sigma^{-1}(2)},\ldots,x_{\sigma^{-1}(n)}). \tag{4}$$

□□□□□σ□□□□$\boldsymbol{\sigma}$□□□□□.

□□2□□□□

$$f(x_1,x_2)=3x_1^2+3x_2^2+4x_1x_2,$$

□x_1,x_2□□□□□□□□□□$3x_2^2+3x_1^2+4x_2x_1$,□□□□□□□□$f(x_1,x_2)$□□.□□□□□□□□□□□□□□.

□□ **2** □$f(x_1,x_2,\ldots,x_n)\in K[x_1,x_2,\ldots,x_n]$,□□□□□$\sigma\in S_n$□□$\sigma f=f$,□□$f(x_1,x_2,\ldots,x_n)$□□□$K$□□□□$n$□□□□□□.

□□□2□□,□□□□n□□□□□□$f(x_1,x_2,\ldots,x_n)$□□□□$ax_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}$,□□□□□□□

$$ax_{\sigma^{-1}(1)}^{i_1}x_{\sigma^{-1}(2)}^{i_2}\cdots x_{\sigma^{-1}(n)}^{i_n},\qquad \forall\sigma\in S_n,$$

□□□σ□□□□n□□□□,$\sigma^{-1}(1)\sigma^{-1}(2)\cdots\sigma^{-1}(n)$□□□$1,2,\ldots,n$□□□$n$□□□□,□□$f(x_1,x_2,\ldots,x_n)$□□□□□□

$$ax_{j_1}^{i_1}x_{j_2}^{i_2}\cdots x_{j_n}^{i_n}$$

□□,□□$j_1j_2\cdots j_n$□$1,2,\ldots,n$□□□n□□□.

□□,□3□□□□□□$f(x_1,x_2,x_3)$□□□□$x_1^2x_2$,□$f(x_1,x_2,x_3)$□

□□□□□□□ $x_{j_1}^2 x_{j_2} x_{j_3}^0$ □□，□□ $j_1 j_2 j_3$ □ 1, 2, 3 □□□ 3 □□□.□□ $f(x_1, x_2, x_3)$ □□□□□□□□□:

$$x_1^2 x_3, \quad x_2^2 x_1, \quad x_2^2 x_3, \quad x_3^2 x_1, \quad x_3^2 x_2.$$

□□ 3 □□□□□□□，□□ $x_1^2 x_2$ □□□□□□□□□□□□□

$$x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_1 + x_2^2 x_3 + x_3^2 x_1 + x_3^2 x_2.$$

□□，□□□□□□□□□□□□□□□□□.

□□□□□□□ $K[x_1, x_2, \dots, x_n]$ □□□□□□□□□□□□□ W □□□.

□□ 1 W □ $K[x_1, x_2, \dots, x_n]$ □□□□□.

□□ □□ W □□□.□□□ W □□□，□□□ W □□□□□□□□.□□ f, g □ W，□□□□□ σ □ S_n，□ $\sigma f = f, \sigma g = g$.□□ σ □□□□□□□□□□□，□□□

$$\sigma(f - g) = \sigma f - \sigma g = f - g,$$
$$\sigma(fg) = (\sigma f)(\sigma g) = fg.$$

□□ $f - g, fg$ □ W.□□ W □ $K[x_1, x_2, \dots, x_n]$ □□□□□. x

□□ 1 □□，$K[x_1, x_2, \dots, x_n]$ □□□□□□□□□□□□□□□□□□□□.

□□□□□□□□□□□□□□ W □□□.

□□ 2 □ $f_1, f_2, \dots, f_n$ □ W，□□□ $K[x_1, x_2, \dots, x_n]$ □□□□□□□ $g(x_1, x_2, \dots, x_n)$，□□

$$h(x_1, x_2, \dots, x_n) \overset{\text{def}}{=} g(f_1, f_2, \dots, f_n)$$

□□□□□□.

□□ □

$$g(x_1, x_2, \dots, x_n) = \sum_{i_1, i_2, \dots, i_n} a_{i_1 i_2 \cdots i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n},$$

□

$$\begin{aligned} h(x_1, x_2, \dots, x_n) &\overset{\text{def}}{=} g(f_1, f_2, \dots, f_n) \\ &= \sum_{i_1, i_2, \dots, i_n} a_{i_1 i_2 \cdots i_n} f_1^{i_1} f_2^{i_2} \cdots f_n^{i_n} \in W. \end{aligned}$$ x

□□ 2 □□，□□□□□ $K[x_1, x_2, \dots, x_n]$ □□ n □□□□□□ $f_1, f_2, \dots, f_n$，□□□□□□□□□□□□□□□□□□□□□，□□□□:□□□□□ n □□□□□□□□□□，□□ $K[x_1, x_2, \dots, x_n]$ □□□□□□□□□□□□□□□□□□□□□ 2 □□□□□?□□□□ K □□□□□□ 1 □□□□□□ $f(x)$ □ n □□

□ $c_1, c_2, \dots, c_n$ □ $f(x)$ □□□□ n □□□□□, □□□□□□□□□ c_1, $c_2, \dots, c_n$ □□□. □□□□□□, □□□□ n □ n □□□□:

$$\sigma_1(x_1, x_2, \dots, x_n) = \sum_{j=1}^{n} x_j,$$

$$\sigma_2(x_1, x_2, \dots, x_n) = \sum_{1 \le j_1 < j_2 \le n} x_{j_1} x_{j_2},$$

$$\cdots\cdots\cdots\cdots \tag{5}$$

$$\sigma_k(x_1, x_2, \dots, x_n) = \sum_{1 \le j_1 < j_2 < \dots < j_k \le n} x_{j_1} x_{j_2} \dots x_{j_k},$$

$$\cdots\cdots\cdots\cdots$$

$$\sigma_n(x_1, x_2, \dots, x_n) = x_1 x_2 \dots x_n.$$

□ n □□□□□□□□□□□, □□□□: □□□ n □□□□□□□□□□ $x_1 x_2 \dots x_k$, □□□□□□□□□□

$$x_{j_1} x_{j_2} \dots x_{j_k} x_{j_{k+1}}^0 \dots x_{j_n}^0$$

□□, □□ $j_1 j_2 \dots j_k j_{k+1} \dots j_n$ □ $1, 2, \dots, n$ □□□ n □□□.

□□ $\sigma_k(x_1, x_2, \dots, x_n) = \sum_{1 \le j_1 < \dots < j_k \le n} x_{j_1} x_{j_2} \dots x_{j_k}$ □□□□□□, $k = 1, 2, \dots, n$.

$\sigma_k(x_1, x_2, \dots, x_n)$, $k = 1, 2, \dots, n$, □□□□□□□□□. □□□□□□□□□ n □□□□□□□□□□. □, □□□□□□□□□ W □□□□□□□□:

□□ 3(□□□□□□□□□) □□ $K[x_1, x_2, \dots, x_n]$ □□□□□□□□□□ $f(x_1, x_2, \dots, x_n)$, □□□ $K[x_1, x_2, \dots, x_n]$ □□□□□□□□□ $g(x_1, x_2, \dots, x_n)$, □□

$$f(x_1, x_2, \dots, x_n) = g(\sigma_1, \sigma_2, \dots, \sigma_n).$$

□□ □□□. □□□□□□□. □□□□□□ $f(x_1, x_2, \dots, x_n)$ □□□□

$$a x_1^{l_1} x_2^{l_2} \dots x_n^{l_n}, \quad a \ne 0. \tag{6}$$

□□□□ $l_1 \ge l_2 \ge \dots \ge l_n$. □□□□: □□ $l_i < l_{i+1}$, □□ $f(x_1, x_2, \dots, x_n)$ □□□□□□, □□□□ $\tau = (i, i+1)$ □□□□ τ □□ $\tau f = f$. □□□ (6) □ τ □□□

$$a x_1^{l_1} x_2^{l_2} \dots x_{i+1}^{l_i} x_i^{l_{i+1}} \dots x_n^{l_n} \tag{7}$$

□□ f □□□. □□ (7) □□□□□ $(l_1, \dots, l_{i-1}, l_{i+1}, l_i, \dots, l_n)$ □□□ (6) □□□□□ $(l_1, \dots, l_{i-1}, l_i, l_{i+1}, \dots, l_n)$, □□□ (6) □ $f(x_1, x_2, \dots, x_n)$ □□□□□. □□ $l_1 \ge l_2 \ge \dots \ge l_n$.

□□□□ $f(x_1, x_2, \dots, x_n)$ □□□, □□□□□□ $\sigma_1, \sigma_2, \dots, \sigma_n$, □□□□

□□

$$\varphi_1(x_1,x_2,\dots,x_n)=a\sigma_1^{l_1-l_2}\sigma_2^{l_2-l_3}\cdots\sigma_{n-1}^{l_{n-1}-l_n}\sigma_n^{l_n},\tag{8}$$

□□ $K[x_1,x_2,\dots,x_n]$ □□□□□□□□□□□□□□□□，□□ $\varphi_1(x_1,$ $x_2,\dots,x_n)$ □□□□□□. □□□□□□□□□□□□□□□□□□□□□□，□□ $\varphi_1(x_1,x_2,\dots,x_n)$ □□□□

$$ax_1^{l_1-l_2}(x_1x_2)^{l_2-l_3}\cdots(x_1x_2\cdots x_{n-1})^{l_{n-1}-l_n}(x_1x_2\cdots x_{n-1}x_n)^{l_n}$$

$$=ax_1^{l_1}x_2^{l_2}\cdots x_{n-1}^{l_{n-1}}x_n^{l_n},$$

□□□ $f(x_1,x_2,\dots,x_n)$ □□□. □

$$f_1(x_1,x_2,\dots,x_n)$$

$$=f(x_1,x_2,\dots,x_n)-\varphi_1(x_1,x_2,\dots,x_n),\tag{9}$$

□ f_1 □□□“□□”f □□□(□ f □□□□□□□□□□ f_1 □□□□□□□□), □□□□□□□□□□□□□□□□□□，□□ $f_1(x_1,x_2,\dots,x_n)$ □ $K[x_1,x_2,\dots,x_n]$ □□□□□□□.

□ $f_1(x_1,x_2,\dots,x_n)$ □□□□□□，□□□□□ $K[x_1,x_2,\dots,x_n]$ □□□□□□□□□

$$f_2(x_1,x_2,\dots,x_n)=f_1(x_1,x_2,\dots,x_n)-\varphi_2(x_1,x_2,\dots,x_n),$$

□□ $\varphi_2(x_1,x_2,\dots,x_n)$ □ $\sigma_1,\sigma_2,\dots,\sigma_n$ □□□□□□□□，□□□□□□□ f_1 □□□□□(□□□(8))，□ f_2 □□□“□□”f_1 □□□.

□□□□□，□□□□ $K[x_1,x_2,\dots,x_n]$ □□□□□□□□□□

$$f,\quad f_1=f-\varphi_1,\quad f_2=f_1-\varphi_2,\quad \dots\tag{10}$$

□□□□□□□□□□□□□□□□□“□”，□□ φ_i □ $\sigma_1,\sigma_2,\dots,\sigma_n$ □□□□□□□，□□□□□□□ f_{i-1} □□□□□. □ f_i □□□□□□□□□□$(p_1,p_2,\dots,$ $p_n)$，□

$$(l_1,l_2,\dots,l_n)>(p_1,p_2,\dots,p_n),$$

□□ l_1 □ p_1. □□□ f_i □□□□□□，□□ p_1 □ p_2 □ … □ p_n. □□ l_1 □ p_1 □ p_2 □ … □ p_n. □□□□□□□□□□□□$(p_1,p_2,\dots,p_n)$□□□□□□，□□□□(10)□□□□□□□ f_i □□□. □□□□□ s，□□ $f_s=0$. □□□

$$f_1=f-\varphi_1,\quad f_2=f_1-\varphi_2,\quad \dots,$$

$$f_{s-1}=f_{s-2}-\varphi_{s-1},\quad f_s=f_{s-1}-\varphi_s=0,$$

□□□□

$$f=\varphi_1+\varphi_2+\dots+\varphi_s.\tag{11}$$

□ $\varphi_i(x_1,x_2,\dots,x_n)=a\sigma_1^{t_{i1}}\sigma_2^{t_{i2}}\cdots\sigma_n^{t_{in}}$，□

$$g(x_1,x_2,\dots,x_n)=\sum_{i=1}^{s}ax_1^{t_{i1}}x_2^{t_{i2}}\cdots x_n^{t_{in}}\in K[x_1,x_2,\dots,x_n],\tag{12}$$

□ $$g(\sigma_1,\sigma_2,\ldots,\sigma_n)=\sum_{i=1}^{s} a\sigma_1^{t_{i1}}\sigma_2^{t_{i2}}\cdots\sigma_n^{t_{in}}=\sum_{i=1}^{s}\varphi_i(x_1,x_2,\ldots,x_n)$$
$$=f(x_1,x_2,\ldots,x_n).$$

□□□□□.

*□□□. □□ $K[x_1,x_2,\ldots,x_n]$ □□□□□□□□□ $g_1(x_1,x_2,\ldots,x_n)$, $g_2(x_1,x_2,\ldots,x_n)$, □□

$$f(x_1,x_2,\ldots,x_n)=g_1(\sigma_1,\sigma_2,\ldots,\sigma_n),$$
$$f(x_1,x_2,\ldots,x_n)=g_2(\sigma_1,\sigma_2,\ldots,\sigma_n),$$

□ $$g_1(\sigma_1,\sigma_2,\ldots,\sigma_n)-g_2(\sigma_1,\sigma_2,\ldots,\sigma_n)=0.$$

□ $$g(x_1,x_2,\ldots,x_n)$$
$$=g_1(x_1,x_2,\ldots,x_n)-g_2(x_1,x_2,\ldots,x_n), \tag{13}$$

□ $$g(\sigma_1,\sigma_2,\ldots,\sigma_n)$$
$$=g_1(\sigma_1,\sigma_2,\ldots,\sigma_n)-g_2(\sigma_1,\sigma_2,\ldots,\sigma_n)=0. \tag{14}$$

□□□□□(13)□□

$$g(x_1,x_2,\ldots,x_n)\ □\ 0,$$

□□□§ 9□□□ 1□, □□ $b_1,b_2,\ldots,b_n$ □ K, □□

$$g(b_1,b_2,\ldots,b_n)\ □\ 0. \tag{15}$$

□ $$\varphi(x)=x^n-b_1x^{n-1}+\cdots+(-1)^kb_kx^{n-k}+\cdots+(-1)^nb_n,$$

□ $\varphi(x)$ □ n □□□□ $\alpha_1,\alpha_2,\ldots,\alpha_n$, □□ Vieta □□□□

$$b_1=\sigma_1(\alpha_1,\alpha_2,\ldots,\alpha_n),\ldots,b_k=\sigma_k(\alpha_1,\alpha_2,\ldots,\alpha_n),\ldots,b_n=\sigma_n(\alpha_1,\alpha_2,\ldots,\alpha_n), \tag{16}$$

$x_1,x_2,\ldots,x_n$ □ $\alpha_1,\alpha_2,\ldots,\alpha_n$ □□, □(14)□□

$$g(\sigma_1(\alpha_1,\ldots,\alpha_n),\ldots,\sigma_n(\alpha_1,\ldots,\alpha_n))=0,$$

□ $$g(b_1,b_2,\ldots,b_n)=0.$$

□□. □□□□□. X

□□□□□□□□□□□□□□ $K[x_1,x_2,\ldots,x_n]$ □□□□□□□□□□ □□□ W□□□□□. □□□□□□□□□□□□□□, □□□□□□□□□ □□□□ $g(x_1,x_2,\ldots,x_n)$, □□

$$f(x_1,x_2,\ldots,x_n)=g(\sigma_1,\sigma_2,\ldots,\sigma_n).$$

□ **1** □ $K[x_1,x_2,x_3]$ □, □□□□□□□□□□□□□□□

$$f(x_1,x_2,x_3)=x_1^2x_2^2+x_1^2x_3^2+x_2^2x_3^2.$$

□ $f(x_1,x_2,x_3)$ □□□□ $x_1^2x_2^2$, □□□□□□□ $(2,2,0)$. □□□□

$$\varphi_1(x_1,x_2,x_3)=\sigma_1^{2-2}\sigma_2^{2-0}\sigma_3^0=\sigma_2^2,$$

□

$$f_1(x_1,x_2,x_3)$$
$$=f(x_1,x_2,x_3)-\varphi_1(x_1,x_2,x_3)$$
$$=x_1^2x_2^2+x_1^2x_3^2+x_2^2x_3^2-(x_1x_2+x_1x_3+x_2x_3)^2$$
$$=-2(x_1^2x_2x_3+x_1x_2^2x_3+x_1x_2x_3^2)$$
$$=-2\sigma_1\sigma_3,$$

$f_1(x_1,x_2,x_3)$□□□□ $-2x_1^2x_2x_3$, □□□□□□□$(2,1,1)$. □

$$\varphi_2(x_1,x_2,x_3)=-2\sigma_1^{2-1}\sigma_2^{1-1}\sigma_3^{1}=-2\sigma_1\sigma_3,$$

$$f_2(x_1,x_2,x_3)=f_1(x_1,x_2,x_3)-\varphi_2(x_1,x_2,x_3)=0,$$

□□

$$f(x_1,x_2,x_3)=\sigma_2^2-2\sigma_1\sigma_3.$$

□□□□□□ n □□□□□□ $f(x_1,x_2,\dots,x_n)$, □□□□□□ $g(x_1,x_2,\dots,x_n)$, □□ $f(x_1,x_2,\dots,x_n)=g(\sigma_1,\sigma_2,\dots,\sigma_n)$, □□□□□□□□□□ □. □□□□□□□□□□□□□□.

□ 2 □ $K[x_1,x_2,\dots,x_n]$□, □□□□□□□□□□□□□□□

$$f(x_1,x_2,\dots,x_n)=□\, x_1^2x_2^2,$$

□□□ $x_1^2x_2^2$ □□ $x_1^2x_2^2$ □□ S_n □□□□□□□□□□□□□□□□□.

□ □□□□□□□□□□, □□(10)□□□ f_i □□□□□□□□ $(p_1,p_2,\dots,p_n)$□□□□

$$l_1 □ p_1 □ p_2 □ \dots □ p_n,$$

□□$(l_1,l_2,\dots,l_n)$□ f □□□□□□□□. □□ f □□□ $x_1^2x_2^2$ □□□□□□ $(2,2,0,\dots,0)$. □□$(p_1,p_2,\dots,p_n)$□□□

$$2 □ p_1 □ p_2 □ \dots □ p_n.$$

□□□□□□□□□ $\sigma_1,\sigma_2,\dots,\sigma_n$ □□□□□□, □□□□□□□□ □□□□□□, □□□□□□□□□□ φ_i □□□□□□□. □□ φ_1 □□□ □□ f □□□ $x_1^2x_2^2$, □□ φ_1 □4□□□□□□. □□□ f □□□4□□□□ □. □□ $f_1=f-\varphi_1$ □□4□□□□□□. □□ $f_2=f_1-\varphi_2$ □□4□□□□ □. □□□□, □□□□, □□(10)□□□□ f_i □□4□□□□□□. □□□□ □

$$p_1+p_2+\dots+p_n=4.$$

□□□□□□, □□□□□□□□□□□□□□:

$$(2,2,0,\dots,0),\quad (2,1,1,0,\dots,0),\quad (1,1,1,1,0,\dots,0),$$

□□□□□ f, f_1, f_2 □□□□□□□□□. □ φ_i □□□□□□

$$\varphi_1=\sigma_1^{2-2}\sigma_2^{2-0}\sigma_3^{0-0}\dots\sigma_{n-1}^{0-0}\sigma_n^{0}=\sigma_2^2,$$

$$\varphi_2=a\sigma_1^{2-1}\sigma_2^{1-1}\sigma_3^{1-0}\sigma_4^{0-0}\dots\sigma_{n-1}^{0-0}\sigma_n^{0}=a\sigma_1\sigma_3,$$

$$\varphi_3=b\sigma_1^{1-1}\sigma_2^{1-1}\sigma_3^{1-1}\sigma_4^{1-0}\sigma_5^{0-0}\dots\sigma_{n-1}^{0-0}\sigma_n^{0}=b\sigma_4,$$

□□ a, b □□□□□□. □□□□□□□□, □

$$f(x_1, x_2, \ldots, x_n) = \sigma_2^2 + a\sigma_1\sigma_3 + b\sigma_4. \tag{17}$$

□□□□ a, b, □□ $x_1, x_2, x_3, x_4, \ldots, x_n$ □ $1, 1, 1, 0, \ldots, 0$ □□, f □□□ 3, $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ □□□□□ $3, 3, 1, 0$, □□□ (17) □□□

$$3 = 3^2 + a \cdot 3 \cdot 1 + b \cdot 0,$$

□□ $a = -2$. □□ (17) □□□

$$f(x_1, x_2, \ldots, x_n) = \sigma_2^2 - 2\sigma_1\sigma_3 + b\sigma_4. \tag{18}$$

$x_1, x_2, x_3, x_4, \ldots, x_n$ □ $1, 1, 1, 1, 0, \ldots, 0$ □□, □ (18) □□

$$6 = 6^2 - 2 \cdot 4 \cdot 4 + b \cdot 1,$$

□□ $b = 2$. □□

$$f(x_1, x_2, \ldots, x_n) = \sigma_2^2 - 2\sigma_1\sigma_3 + 2\sigma_4.$$

□ 2 □□□□□□□ $f(x_1, x_2, \ldots, x_n)$ □□□□□. □□□□□□□□□□ f □□□□□, □□□ f □□□□□□□□□□□. □□□□□□□□(□□□□□□□□□)□□□ 2 □□□, □□□□□□□□□□□□.

□□□□□□□□□□□□□□□□□, □□□□ K □□□□□□□□□□□□□□.

□□□ K □□□□□ 1 □□□□

$$f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

□□□□□□ n □□□ $c_1, c_2, \ldots, c_n$. □□□□

$$f(x) \text{ □□□□□□□□}$$

$$\Leftrightarrow D(c_1, c_2, \ldots, c_n) \overset{\text{def}}{=} \prod_{1 \leq j < i \leq n} (c_i - c_j)^2 = 0.$$

□□ $f(x)$ □ n □□□ $c_1, c_2, \ldots, c_n$ □□□□□, □□□□ $f(x)$ □□□□□□ $D(c_1, c_2, \ldots, c_n)$. □ Vieta □□□

$$\begin{aligned} -a_{n-1} &= c_1 + c_2 + \ldots + c_n = \sigma_1(c_1, c_2, \ldots, c_n), \\ a_{n-2} &= \sum_{1 \leq j_1 < j_2 \leq n} c_{j_1} c_{j_2} = \sigma_2(c_1, c_2, \ldots, c_n), \\ &\ldots\ldots\ldots\ldots \\ (-1)^n a_0 &= c_1 c_2 \ldots c_n = \sigma_n(c_1, c_2, \ldots, c_n). \end{aligned} \tag{19}$$

□□□□□□, □□ $D(c_1, c_2, \ldots, c_n)$ □□□ $\sigma_1(c_1, c_2, \ldots, c_n), \sigma_2(c_1, c_2, \ldots, c_n), \ldots, \sigma_n(c_1, c_2, \ldots, c_n)$ □□□□, □□□□□□□ $f(x)$ □□□ $a_{n-1}, a_{n-2}, \ldots, a_0$ □□□□. □□□ $D(c_1, c_2, \ldots, c_n) = \prod_{1 \leq j < i \leq n} (c_i - c_j)^2$ □□□ $c_1, c_2, \ldots, c_n$ □□□□□, □□□□□□□□□□□□□□□□□□.

□□□□ K □ n □□□□□□

$$D(x_1, x_2, \ldots, x_n) = \prod_{1 \leq j < i \leq n} (x_i - x_j)^2. \tag{20}$$

□□□□□□□□□□□,□□ $K[x_1, x_2, \ldots, x_n]$□□□□□□□□□ $g(x_1, x_2, \ldots, x_n)$,□□

$$D(x_1, x_2, \ldots, x_n) = g(\sigma_1, \sigma_2, \ldots, \sigma_n). \tag{21}$$

□□□ $x_1, x_2, \ldots, x_n$ □□□ $c_1, c_2, \ldots, c_n$ □□,□(19)□(21)□□

$$D(c_1, c_2, \ldots, c_n) = g(-a_{n-1}, a_{n-2}, \ldots, (-1)^n a_0). \tag{22}$$

□□□□

□□ 4　□□ K□□□□□□1□ n□□□□

$$f(x) = x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

□□□□□□□□□□□□□□□□

$$g(-a_{n-1}, a_{n-2}, \ldots, (-1)^n a_0) = 0. \tag{23}$$

□□□ $f(x)$□□□ $a_{n-1}, a_{n-2}, \ldots, a_0$ □□□□　X

$$g(-a_{n-1}, a_{n-2}, \ldots, (-1)^n a_0)$$

□□ $f(x)$□□□□,□□ $D(f)$.□□□□□□□ $f(x)$□□□□□□□□□.

□□□□ $f(x)$□□□□ $D(f)$□ ?□(22)□□

$$\begin{aligned} D(f) &= g(-a_{n-1}, a_{n-2}, \ldots, (-1)^n a_0) = D(c_1, c_2, \ldots, c_n) \\ &= \prod_{1 \leq j < i \leq n} (c_i - c_j)^2. \end{aligned} \tag{24}$$

□□□ $\prod_{1 \leq j < i \leq n} (c_i - c_j)^2$ □□□□□□□□□□□,□

$$B = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ c_1 & c_2 & \ldots & c_n \\ \ldots & \ldots & & \ldots \\ c_1^{n-1} & c_2^{n-1} & \ldots & c_n^{n-1} \end{pmatrix}, \tag{25}$$

□ $|B| = \prod_{1 \leq j < i \leq n} (c_i - c_j)$.□□□ $|B'| = |B|$,□□□(24)□□

$$D(f) = \prod_{1 \leq j < i \leq n} (c_i - c_j)^2 = |B||B'| = |BB'|. \tag{26}$$

□□

$$D(f) = \left| \begin{matrix} 1 & 1 & \ldots & 1 \\ c_1 & c_2 & \ldots & c_n \\ \ldots & \ldots & & \ldots \\ c_1^{n-1} & c_2^{n-1} & \ldots & c_n^{n-1} \end{matrix} \quad \begin{matrix} 1 & c_1 & \ldots & c_1^{n-1} \\ 1 & c_2 & \ldots & c_2^{n-1} \\ \ldots & \ldots & & \ldots \\ 1 & c_n & \ldots & c_n^{n-1} \end{matrix} \right|$$

$$
= \begin{vmatrix} n & \sum_{i=1}^n c_i & \cdots & \sum_{i=1}^n c_i^{n-1} \\ \sum_{i=1}^n c_i & \sum_{i=1}^n c_i^2 & \cdots & \sum_{i=1}^n c_i^n \\ \cdots & \cdots & & \cdots \\ \sum_{i=1}^n c_i^{n-1} & \sum_{i=1}^n c_i^n & \cdots & \sum_{i=1}^n c_i^{2n-2} \end{vmatrix}. \tag{27}
$$

(27) [illegible], [illegible] $D(f)$, [illegible]

$$
\sum_{i=1}^n c_i^k, \quad k=0,1,2,\dots,2n-2. \tag{28}
$$

[illegible] $f(x)$ [illegible] n [illegible] $c_1, c_2, \dots, c_n$ [illegible], [illegible] $f(x)$ [illegible] c_i^k. [illegible] c_i^k [illegible] $c_1, c_2, \dots, c_n$ [illegible], [illegible]. [illegible] n [illegible]

$$
\begin{aligned} & s_k(x_1, x_2, \dots, x_n) \\ = & x_1^k + x_2^k + \dots + x_n^k, \quad k=0,1,2,\dots \end{aligned} \tag{29}
$$

[illegible] n [illegible].

[illegible], [illegible] s_k [illegible]. [illegible].

[illegible] (Newton) [illegible]* [illegible] 1 [illegible] k [illegible] n [illegible],

$$
s_k - \sigma_1 s_{k-1} + \sigma_2 s_{k-2} + \dots + (-1)^{k-1}\sigma_{k-1} s_1 + (-1)^k k\sigma_k = 0; \tag{30}
$$

[illegible] $k > n$ [illegible],

$$
s_k - \sigma_1 s_{k-1} + \sigma_2 s_{k-2} + \dots + (-1)^{n-1}\sigma_{n-1} s_{k-n+1} + (-1)^n \sigma_n s_{k-n} = 0. \tag{31}
$$

[illegible], [illegible] $s_{k-1}(c_1, \dots, c_n), \dots, s_1(c_1, \dots, c_n)$ [illegible] $\sigma_1(c_1, \dots, c_n), \dots, \sigma_n(c_1, \dots, c_n)$ [illegible] $s_k(c_1, \dots, c_n)$. [illegible] (27) [illegible] $f(x)$ [illegible] $D(f)$.

$f(x)$ [illegible] $D(f)$ [illegible] $f(x)=0$ [illegible].

[illegible]: [illegible], $f(x)$ [illegible] 1. [illegible] $f(x)$ [illegible] a_n, [illegible] $a_n^{-1} f(x)$ [illegible] $D(a_n^{-1} f)$, [illegible] $f(x)$ [illegible]

$$
D(f) := a_n^{2n-2} D(a_n^{-1} f). \tag{32}
$$

[illegible] 3 [illegible] K [illegible] $f(x) = x^2 + bx + c = 0$ [illegible].

---

* [illegible] ([illegible] 1996 [illegible] 12 [illegible] 1 [illegible]) [illegible] 125 [illegible] 127 [illegible].

[illegible] $f(x)$ [illegible] c_1, c_2, [illegible]

$$\sigma_1(c_1, c_2) = -b, \quad \sigma_2(c_1, c_2) = c.$$

[illegible]

$$s_1 = \sigma_1, \quad s_2 = \sigma_1 s_1 - 2\sigma_2 = \sigma_1^2 - 2\sigma_2,$$

[illegible]
$$s_1(c_1, c_2) = -b, \quad s_2(c_1, c_2) = b^2 - 2c,$$

[illegible]
$$D(f) = \begin{vmatrix} 2 & -b \\ -b & b^2 - 2c \end{vmatrix} = b^2 - 4c. \tag{33}$$

[illegible] $b^2 - 4c$ [illegible] $x^2 + bx + c = 0$ [illegible], [illegible] "[illegible]" [illegible].

[illegible] 4 [illegible] K [illegible]

$$f(x) = x^3 + ax + b = 0 \tag{34}$$

[illegible].

[illegible] $f(x)$ [illegible] c_1, c_2, c_3, [illegible]

$$\sigma_1(c_1, c_2, c_3) = 0, \quad \sigma_2(c_1, c_2, c_3) = a, \quad \sigma_3(c_1, c_2, c_3) = -b.$$

[illegible]

$$s_1(c_1, c_2, c_3) = \sigma_1(c_1, c_2, c_3) = 0, \quad s_2(c_1, c_2, c_3) = -2a,$$

$$s_3(c_1, c_2, c_3) = -3b, \quad s_4(c_1, c_2, c_3) = 2a^2,$$

[illegible]

$$D(f) = \begin{vmatrix} 3 & 0 & -2a \\ 0 & -2a & -3b \\ -2a & -3b & 2a^2 \end{vmatrix} = -4a^3 - 27b^2. \tag{35}$$

* [illegible]

$$f(x) = x^3 + a_1 x^2 + a_2 x + a_3 = 0, \tag{36}$$

[illegible] $D(f)$ [illegible] (35) [illegible]. [illegible]: x [illegible] $x - \frac{a_1}{3}$ [illegible], [illegible]

$$g(x) := f\left(x - \frac{a_1}{3}\right),$$

[illegible] $f\left(x - \frac{a_1}{3}\right)$ [illegible], $g(x)$ [illegible] (x^2 [illegible] 0). [illegible] $g(x)$ [illegible] c_0, [illegible]

$$f\left(c_0 - \frac{a_1}{3}\right) = g(c_0) = 0,$$

[illegible] $c_0 - \frac{a_1}{3}$ [illegible] $f(x)$ [illegible]. [illegible] $g(x)$ [illegible], [illegible] $f(x)$ [illegible]

□□□.□□□□□□□,□□□□□□ x^2 □□□ $a_1 = 0$.

□ □ 7.10

1. □ $f(x_1, x_2, x_3)$□□□ K □□□□□□□□□:

$$f(x_1, x_2, x_3) = x_1^3 x_2^2 + x_1^3 x_3^2 + x_1^2 x_2^3 + x_1^2 x_3^3 + x_2^3 x_3^2 + x_2^2 x_3^3,$$

□□ $f(x_1, x_2, x_3)$□□□□□□.

2. □ $K[x_1, x_2, x_3]$□□□□ $x_1^3 x_2$ □□□□□□□,□□□□□□□□□□□□□□.

3. □ $K[x_1, x_2, x_3]$□,□□□□□□□□□□□□□□□□□:

(1) $x_1^3 x_2 + x_1 x_2^3 + x_1^3 x_3 + x_1 x_3^3 + x_2^3 x_3 + x_2 x_3^3$;

(2) $x_1^4 + x_2^4 + x_3^4$;

(3) $(x_1 x_2 + x_3^2)(x_2 x_3 + x_1^2)(x_3 x_1 + x_2^2)$.

4. □ $K[x_1, \ldots, x_n]$□,□□□□□□□□□□□□□□□□□(n□3):

(1) □ x_1^3; (2) □ $x_1^2 x_2^2 x_3$.

5. □ $K[x_1, x_2, x_3]$□,□□□□□□□□□ s_2, s_3, s_4 □□□□□□□□□□.

6. □ $f(x)$□□□□□□□□□,□□:□ $D(f) = 0$□,$f(x)$□ C □□□□,□□ $f(x)$ □ C □□□□□□;□ $D(f) > 0$□,$f(x)$□□□□□□□□□□;□ $D(f) < 0$□,$f(x)$□□□□,□□□□□□.

7. □ $f(x)$□ $K[x]$,□□ $f(x)$□□□□□□ 1,a□ K,$g(x) = (x - a) f(x)$,□□:

$$D(g) = D(f) f(a)^2.$$

8. □ $f(x) = x^n + a$□ $K[x]$□□□□.

9. □□□ K □□□□□□□

$$f(x) = x^3 + a_1 x^2 + a_2 x + a_3 = 0$$

□□□□.

§ 11 □ □ □

□□□□□□,Z, $M_n(K)$, $K[x]$, $K[x_1, x_2, \ldots, x_n]$□□□.□□□□□□□□□□□,□□□□□□□.□□□□□□□□□□□□□□□?

□□□□□□□ 4,□□ Z □□□□□:

$$\bar{0} \overset{\text{def}}{=} \{4k \mid k \in Z\}, \quad \bar{1} \overset{\text{def}}{=} \{4k + 1 \mid k \in Z\},$$

$$\bar{2} \overset{\text{def}}{=} \{4k + 2 \mid k \in Z\}, \quad \bar{3} \overset{\text{def}}{=} \{4k + 3 \mid k \in Z\}.$$

□□□□□ 4 □□□.$\bar{2}$□□ 2 □□□□,$\bar{2}$□□□□□□□□□□□□□□.□□

□□.□

$$Z_4 \stackrel{\text{def}}{=} \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}. \tag{1}$$

□ Z_4 □□□□□□□□□□□□:

$$\bar{i}+\bar{j} \stackrel{\text{def}}{=} \overline{i+j}, \quad \bar{i}\cdot\bar{j} \stackrel{\text{def}}{=} \overline{ij}. \tag{2}$$

□□□□(2)□□□□□□□□,□□□□□□□□□□□□□.□ $\bar{i}=\bar{a}, \bar{j}=\bar{b}$,
□

$$i-a=4k, \quad j-b=4l.$$

□□

$$(i+j)-(a+b)=(i-a)+(j-b)=4(k+l),$$
$$ij-ab=ij-aj+aj-ab=(i-a)j+a(j-b)$$
$$=4kj+4la=4(kj+la).$$

□□ $$\overline{i+j}=\overline{a+b}, \quad \overline{ij}=\overline{ab}.$$

□□□□(2)□□□□□□□□.□□

$$\bar{2}+\bar{3}=\bar{5}=\bar{1}, \quad \bar{2}\cdot\bar{3}=\bar{6}=\bar{2}, \quad \bar{2}\cdot\bar{2}=\bar{4}=\bar{0}.$$

□□□□,$\bar{0}$□ Z_4 □□□□□□□□,$\bar{1}$□ Z_4 □□□□□□□,□□ Z_4 □□□□□□□□□□□,□□□□□□□□□ $\bar{2}$. Z_4 □□□ 4 □□□□□.□ □□□□□□□□□□□□□□.

□□□,□□□□□□□ m,□□ Z □□□

$$\bar{i} \stackrel{\text{def}}{=} \{km+i \mid k\in Z\}, \quad i=0,1,2,\dots,m-1,$$

□ $\bar{i}$ □□ m □□□,i □□□□,$\bar{i}$ □□□□□□□□□□□□□□.□□□□, □□ $a\in Z$,□

$$a\in\bar{i} \Leftrightarrow m\mid a-i. \tag{3}$$

□(3)□□□□□,□□□ Z □□□□□□□□□□□:

$$a\equiv b \stackrel{\text{def}}{\Longleftrightarrow} m\mid a-b \tag{4}$$

□□□□,□□□□□□□□□□□□□□□.□□□□□□□□□□.□□□□ m□□□□,□□□ a□b□□

$$a\equiv b \pmod{m}, \tag{5}$$

□□“a□b□m□□”.□□,□□ m□□□□□□□□□□□□ m□□□. □□□ Z □□□ m □□□□□□□(□□□□□□□□□)□□ Z/mZ,□ $Z/(m)$,□ Z_m.□□□

$$Z_m=\{\bar{0},\bar{1},\bar{2},\dots,\overline{m-1}\}. \tag{6}$$

□ Z_m □□□□□□□□□□□□:

$$\bar{i}+\bar{j} \stackrel{\text{def}}{=} \overline{i+j}, \quad \bar{i}\,\bar{j} \stackrel{\text{def}}{=} \overline{ij}. \tag{7}$$

□□□□□□□□(7)□□□□□□□□. □□□□, Z_m □□□□□□□□□□

□□, □□□□ m□□□□. □ Z_m □□□□□□□□□:

$$\bar{i}-\bar{j}\overset{\text{def}}{=}\bar{i}+(-\bar{j}). \tag{8}$$

□□, □ 2□□□□ $Z_2=\{\bar{0},\bar{1}\}$. □□□□□□□ $\bar{1}$□□.

□□, □ 5□□□□ $Z_5=\{\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$. □□

$$\bar{2}\,\bar{3}=\bar{6}=\bar{1},\quad \bar{4}\,\bar{4}=\overline{16}=\bar{1},$$

□□ Z_5 □□□□□□□□□.

□ Z_2, Z_5 □□□□□□□□□□:

□□ 1 □ F □□□□□□□ 1(□0)□□□□, □□ F □□□□□□□□

□□, □□ F □□□□.

□□, □□□□□, □□□□□□□□□□, □□□□□□□.

□□□□□□□□, Z_2, Z_5 □□□, □□□□□□□□□□□□□□□.

Z_4 □□□(□□ $\bar{2}$□□□). □□ Z_4 □ Z_2, Z_5 □□□□□, □□□□□□□

□□:

□□ 1 □□ p □□□□□, □□ p □□□□ Z_p □□□□.

□□ □□□ Z_p □□□□□□□ $\bar{a}$□□, □□ $0<a<p$. □□ p □□□, □ $p\nmid a$, □□ $(p,a)=1$. □□□□ u, v□ Z, □□

$$ua+vp=1.$$

□□□□, $\bar{u}\,\bar{a}+\bar{v}\,\bar{p}=\bar{1}$, □ $\bar{u}\,\bar{a}=\bar{1}$. □□ $\bar{a}$□□. □□ Z_p □□□□. x

Z_p □□□ p □□□□, □□□□ p □□□. □ Z_p □,

$$p\bar{1}=\underbrace{\bar{1}+\bar{1}+\dots+\bar{1}}_{p□}=\bar{p}=\bar{0}. \tag{9}$$

□□ $0<l<p$ □, □□□□, $l\bar{1}$□$\bar{0}$.

□□□ K □, □□□□□□□ n, □□ n1□0.

□□ 2 □ F □□□□, □□□□□□□ e □□□□□□□□ p, □□ $pe=0$, □□□ $0<l<p$ □□ le□0, □□□ F □□□□ p; □□□□□□□□ n, □ ne□0, □□□ F □□□□ 0. □□ F □□□□□ char F.

□□ 2 □ F □□□□, □ F □□□□□□ 0, □□□□□□□.

□□ □ F □□□□□ 0, □□□□□□□□ n, □□ $ne=0$. □ n □□ $ne=0$□□□□□□□□. □□ n □□□□, □ $n=n_1n_2$, □□ $0<n_i<n, i=1,2$. □□

$$(n_1e)(n_2e)=(n_1n_2)e=ne=0. \tag{10}$$

□□ n_1e□0, □ n_1e □□. □(10)□□□□□ $(n_1e)^{-1}$□, $n_2e=0$, □□ n □□□□□. □□ $n_1e=0$, □□ n □□□□□. □□ n □□□□. □□ F □□□□□

□ n. x

□□ 3 □□□ F □□□□□□ p, □

$$ne=0 \Leftrightarrow p \mid n.$$

□□ □□□.□ $p \mid n$, □ $n=lp$. □□

$$ne=(lp)e=l(pe)=0.$$

□□□.□ $ne=0$. □□ $n=hp+r$, 0□ $r<p$, □

$$0=ne=(hp+r)e=hpe+re=re.$$

□□ $\mathrm{char}\,F=p$, □ $r<p$. □□□□□, $r=0$. □□ $n=hp$, □ $p \mid n$. x

□□ 4 □□ F □□□□□□ p, a□ F □ a□ 0, □ $na=0$ □□□□ $p \mid n$.

□□

$$na=0 \Leftrightarrow n(ea)=0 \Leftrightarrow (ne)a=0$$

$$\Leftrightarrow ne=0 \Leftrightarrow p \mid n.$$ x

□□ 4□□□□,□□□□□ p □□ F □,□□□□□□□□:□□ a□ 0, □ $pa=0$.

□□□□□ K □□□□□□(□□□□□),□□□□□□□□ F □□□□□□□(□□□□□□),□□□□ F □□□□□□□ $F[x]$(□ n □□□□□ $F[x_1, x_2, \ldots, x_n]$).□□□□,□□□□ K □□□□□□□ $K[x]$ □□□,□□□□□□□□□□□□□□□□□□□□□□,□□□□□□□ F □□□□□□□□ $F(x)$□□□.□□□□□,□□□ F □□□□□□ p, □ F □□□□□□ p □□□□.

□□,□□□□ K □□□□□□□□□ $f(x)$□ $g(x)$,□□□□□□□,□□□□□□□□□□□□□□ f □ g □□□□.□□□□□□□□□□□□ K □□□□□□□.□□□□□□□□□□□□□□□□□□□□.□□,□ $Z_3[x]$□,□ $f(x)=x^3-x$, $g(x)=0$. □□ $f(x)$□ $g(x)$.□□□ $f(x)$□□□□□□□□ f □□

$$f(\bar{0})=\bar{0},\quad f(\bar{1})=\bar{1}^3-\bar{1}=\bar{0},\quad f(\bar{2})=\bar{2}^3-\bar{2}=\bar{0},$$

□□ f □□□□.□ $g(x)$□□□□□□□□□□.

□□□□ F □□□□□□□□ $F[x]$□,□□□□□□□□□□□□□□□□□□;□□□□□□□□□□□,□□.

□ □ 7.11

1. □□:□ F □□□□□□□□□,□□□□□□□.
2. □□□ 7□□□□ Z_7 □□□□□□□□.

* 3. □

$$F=\left\{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \middle| a, b \in \mathbf{R}\right\},$$

□□:F□□□□□□□□□□□□□□□,□□□ F□□□□□□.

4. □

$$F=\left\{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \middle| a, b \in \mathbf{Z}_3\right\},$$

□□:F□□□□ 9□□□□□,□□ char $F=3$.

5. □□:□□□□ p□□ F□,□□□□:

$$(a+b)^p = a^p + b^p.$$

6. □□ $Z_2[x]$□□□□□□□□□□□□□□□□□.

7. □ $f(x)=x^5-x^2+1\in Z[x]$,□□ $f(x)$□ Q□□□□□□.

8. □ $f(x)=x^4+3x^3+3x^2-5\in Z[x]$,□□ $f(x)$□ Q□□□□□□.

□8□ □□□□

□□□□□□□□□□□□□“□□”□□□□□□.□□□□□□□□□□□□□.□□□□□□□□□□□□□□□□□□□□□□□,□□□□□□□□□□□□8□□□□□.

□□□□□□□□□□“□□”□□□□□□□□□□□□□□.□□□□□□□□□□□ n □□□(□ n □□□□□).□□ K □□□ n □□□□□□□□□□ K^n,□□□□□□□□□,□□□□□□□□□□□□□□8□□□□□.

□□□□□□□□□□□□□□□□□□,□□□□□□□□□□□□□□□□□□.□□□□□□□□□□□□□□□.□□ K □□□ $s\times n$ □□□□□□□□,□□□□□□□□□□□□□□□□□□□□□□□8□□□□□.

□□□□□□□,□□□□□□□□□□□□□□:□□□□,□□□□□□□□□.□□□□□□□□□□□□□□□□□□□.□□□□□□□□□□,□□□□□□,□□□□□□□□□□□□,□□□□□□□□□□□□□□□□□□□□□□□.

§1 □□□□□□□

□□□□□□□□□□□□,n□□□,□□□□□□□□□□□:□□□□,□□□□,□□□□(□□□□□□□),8□□□□□.□□□□□□□□□□□□.

□□ 1 □ V □□□□□□□,F □□□□.□ V □□□□□□□□□□:$(\alpha,\beta)\longmapsto\gamma$,□□□□,□ γ □□ α □ β □□,□□ $\gamma=\alpha+\beta$.□ F □ V □□□□□□□□□,□ $F\times V$ □ V □□□□□:$(k,\alpha)\longmapsto\delta$,□□□□□□(□ F □□□□,□□□□□□),□ δ □□ k □ α □□□□□(□ F □□□□,□□□□□□),□□ $\delta=k\alpha$.□□□□□□□□□□□□□ 8 □□□□□:□□□□ α,β,γ □ V,□□□ k,l □ F,□

1° $\alpha+\beta=\beta+\alpha$ (□□□□□);

2° $(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma)$ (□□□□□);

3° V □□□□□□,□□ 0,□□□

$$\alpha+0=\alpha,\ \forall\alpha\in V,$$

[illegible] 0 [illegible] V [illegible];

4° [illegible] $\alpha\in V$, [illegible] $\beta\in V$, [illegible]

$$\alpha+\beta=0,$$

[illegible] β [illegible] α [illegible];

5° $1\alpha=\alpha$;

6° $(kl)\alpha=k(l\alpha)$;

7° $(k+l)\alpha=k\alpha+l\alpha$;

8° $k(\alpha+\beta)=k\alpha+k\beta$,

[illegible] V [illegible] F [illegible].

[illegible], [illegible], [illegible].
[illegible] F [illegible] V [illegible].

[illegible], [illegible]
[illegible], [illegible] R [illegible].

[illegible] K [illegible] n [illegible] K^n, [illegible]
[illegible], [illegible] K [illegible]. [illegible] 3 [illegible], [illegible] K^n [illegible] K
[illegible] n [illegible].

[illegible] K [illegible] $s\times n$ [illegible], [illegible], [illegible]
[illegible] K [illegible], [illegible] $M_{s\times n}(K)$.

[illegible].

[illegible] 1 [illegible] X [illegible] R [illegible], [illegible] X [illegible]
[illegible] R^X, [illegible], [illegible], [illegible] R [illegible]
[illegible].

[illegible] 2 [illegible] X [illegible], F [illegible], [illegible] X [illegible] F [illegible]
[illegible] X [illegible] F [illegible]. X [illegible] F [illegible] F^X. [illegible] F^X [illegible]
[illegible]: [illegible] $f,g\in F^X$, $k\in F$, [illegible]

$$(f+g)(x)\overset{\mathrm{def}}{=}f(x)+g(x),\ \forall x\in X,$$

$$(kf)(x)\overset{\mathrm{def}}{=}k(f(x)),\ \forall x\in X,$$

[illegible] 8 [illegible]. [illegible] F^X [illegible] F [illegible]
[illegible]. F^X [illegible], [illegible] 0, [illegible]

$$0(x)=0,\ \forall x\in X.$$

[illegible] 3 [illegible] K [illegible] $K[x]$, [illegible],
[illegible] K [illegible], [illegible] K [illegible].

[illegible] 4 [illegible] C [illegible] R [illegible], [illegible]

□□,□□□□□□□□ a□□□ z□□.

□ 5 □□ K□□□□□□□□□□□□,□□□□□□□ K□□□□,□□□□□□ K□□□□.

□□□□□,□□□□□□□□□□□□□□□.□□□□□,□□□□□□□□□□□□□,□□□□□,□□□□□□□□□□□□□□,□□□□□□□□□□□□□□□□□.

□ V□□ F□□□□□□□□.

1. V□□□□□□□□.

□□ □□ $0_1, 0_2$ □ V□□□□□□,□

$$0_1 + 0_2 = 0_1,$$

$$0_1 + 0_2 = 0_2 + 0_1 = 0_2,$$

□□ $0_1 = 0_2$. X

2. V□□□□□ α □□□□□□□□.

□□ □□ β_1, β_2 □□ α □□□□,□

$$(\beta_1 + \alpha) + \beta_2 = (\alpha + \beta_1) + \beta_2 = 0 + \beta_2 = \beta_2 + 0 = \beta_2,$$

$$\beta_1 + (\alpha + \beta_2) = \beta_1 + 0 = \beta_1.$$

□□□□□□□,□ $\beta_1 = \beta_2$. X

□□□ α □□□□□□□□□ $-\alpha$.

□□□□□,□□□ V□□□□□□□:□□ α, β□ V,

$$\alpha - \beta \overset{\text{def}}{=} \alpha + (-\beta).$$

3. $0\alpha = 0$, Aα□ V.

□□ $0\alpha + \alpha = 0\alpha + 1\alpha = (0+1)\alpha = 1\alpha = \alpha$.

□□□□□□$(-\alpha)$,□

$$(0\alpha + \alpha) + (-\alpha) = \alpha + (-\alpha).$$

□□□□□□□□□□□□ 4°, 3°,□

$$0\alpha = 0.$$ X

4. $k0 = 0$, A k□ K.

□□ $$k0 + k0 = k(0 + 0) = k0.$$

□□□□□□ $-k0$,□

$$(k0 + k0) + (-k0) = k0 + (-k0).$$

□□□□□□□□□□□□ 4°, 3°,□

$$k0 = 0.$$ X

5. □□ $k\alpha = 0$, □□ $k = 0$ □ $\alpha = 0$.

□□ □□ k□ 0,□

$$\alpha = 1\alpha = (k^{-1}k)\alpha = k^{-1}(k\alpha) = k^{-1}0 = 0.$$ x

6. $(-1)\alpha = -\alpha$, Aα□ V.

□□ $\alpha + (-1)\alpha = 1\alpha + (-1)\alpha = [1 + (-1)]\alpha = 0\alpha = 0.$

□ α □□□□□□□□,□ $(-1)\alpha = -\alpha$. x

□ $\alpha_1, \alpha_2, \ldots, \alpha_s$ □ V □□□□□□,□□ F □□□□□ $k_1, k_2, \ldots, k_s$,□□ $k_1\alpha_1 + k_2\alpha_2 + \ldots + k_s\alpha_s$ □□ $\alpha_1, \alpha_2, \ldots, \alpha_s$ □□□□□□□,□□ $k_1, k_2, \ldots, k_s$ □□□□.

□□ β□ V,□□□ F □□□□□ $c_1, c_2, \ldots, c_s$,□□

$$\beta = c_1\alpha_1 + c_2\alpha_2 + \ldots + c_s\alpha_s,$$

□□ β □□□ $\alpha_1, \alpha_2, \ldots, \alpha_s$ □□□□.

□□ 2 V □□□□ $\alpha_1, \alpha_2, \ldots, \alpha_s$ (s□1) □□□□□□□□,□□□ F □□□□□□□□ $k_1, k_2, \ldots, k_s$,□□

$$k_1\alpha_1 + k_2\alpha_2 + \ldots + k_s\alpha_s = 0. \tag{1}$$

□□,□□□ $\alpha_1, \alpha_2, \ldots, \alpha_s$ □□□□□□□□. □,□□□

$$k_1\alpha_1 + k_2\alpha_2 + \ldots + k_s\alpha_s = 0.$$

□□□□ $k_1 = k_2 = \ldots = k_s = 0$,□□□□□ $\alpha_1, \alpha_2, \ldots, \alpha_s$ □□□□□□.

□□□□□□ $s = 0$ □□□□□□□□□□□□□,□□□□□□□□□□□□□.

V□□□□□□□□□□. □□□□□□□ W,□□□□□□□□□□□(□□□□□□□□□□□),□□ $\alpha_1, \alpha_2, \ldots, \alpha_s$,□□□□□□□□□. □□□□□□□□□□,□□□□□□□□□□,□□□□□□□□.

V□□□□□□□ W□□□□□□□(□□□□)□,□□□□ W□□□□□□□□(□□□□□□□)□□□□□□□□□□□□(□□□□)□.

□□ 3 □ W□ V□□□□□□□,□□ W□□□□□□□□□□□□□,□□ W□□□□□□;□□ W□□□□□□□□□□□□□□,□□ W □□□□□□.

□ 6 □□□□ α □□□□ � $\alpha = 0$.

□ 7 □□□□□□□□□□□□□□□.

□□□□□□□□ 3□§2□□□□□,□□□□□□□□□□. □□□□□□□,□□□□□□□ 1□□□ 6□□□□□ K□□□□□□□□□□□□□ K□□□□□□(□□□□□□□□)□□□□ K□□□□□ F□□□□□.

□□ 1 □□□□□□□□□ 2□□□□ W□□□□□□□□ W□□□□□□□□□□□□□□□□□□□□□□□□□□.

□□2 □□□□ $\alpha_1,\alpha_2,\ldots,\alpha_s$ □□□□,□□□ β □□□ $\alpha_1,\alpha_2,\ldots,\alpha_s$ □□□□□□□□□□□□ $\alpha_1,\alpha_2,\ldots,\alpha_s,\beta$ □□□□.

□□□□□□□□ 3□§3□□□□□,□□□□□□□□,□□□□□□□□□□.

□□4 □ W_1, W_2 □□ V □□□□□,□□ W_1 □□□□□□□□□□ W_2 □□□□□□□□□□□,□□ W_1 □□□ W_2 □□□□.□□ W_1 □ W_2 □□□□□□□□,□□ W_1 □ W_2 □□□□.

□□□□,□□□□□□□□,□□ V □□□□□□□□□□□□.□□,□□□□□□□□□□□□□□.

□□1 □□□□ $\beta_1,\beta_2,\ldots,\beta_r$ □□□□□□ $\alpha_1,\alpha_2,\ldots,\alpha_s$ □□□□.□□ $r>s$,□□□□□ $\beta_1,\beta_2,\ldots,\beta_r$ □□□□.

□□3 □□□□ $\beta_1,\beta_2,\ldots,\beta_r$ □□□□□□ $\alpha_1,\alpha_2,\ldots,\alpha_s$ □□□□.□□ $\beta_1,\beta_2,\ldots,\beta_r$ □□□□,□ r□ s.

□□4 □□□□□□□□□□□□□□□□□□□□□.

□□5 □□□(□)□□□□□(□□)□□□□□□□□□□(□),□□□□□□□(□□)□□□□□□□□,□□□□□□□□(□)□□□□□(□□□□□□)□□□□□□□□,□□□□□□□□(□□)□□□□□.

□□5 □□□(□)□□□□□□□□□□(□)□□.

□□□5□,□□□(□)□□□□□□□□□□□□(□)□□.□□□

□□6 □□□□□□□□□□□□□□□□□□□□□□□□.

□□6 □□□□□□□□□□□□□□□□□□□□□□□□□□□□□.

□□□□□□□□□□□□□□□.

□□□ $\alpha_1,\alpha_2,\ldots,\alpha_s$ □□□□ $\operatorname{rank}\{\alpha_1,\alpha_2,\ldots,\alpha_s\}$.

□□7 □□□□□□□□□□□□□□□□□□□□□□□□□□□□□.

□□8 □□□□□(Ⅰ)□□□□□(Ⅱ)□□□□,□(Ⅰ)□□□(Ⅱ)□□.

□□9 □□□□□□□□□□□.

□□□□□□□□□□□□□(3□□□□□□□)□□.

□□ K□ n□□□□□□□□□□□□□□□□□□□□□(□□□□□□)□□.

□□ K□ n□□□□□ K^n □□□□□□□□ U□□□□ U□□□□□□.

□□□□□□□,□□□ F□□□□□□□ V□□□,□ V□□□□□□ □□□□□.

□ V□□ F□□□□□□□□.

□□ 7 V□□□□□ S□□□□□□□□□□:

1° □□□ S□□□□□□;

2° V□□□□□□□□□ S□□□□□□□□□□□,

□□ S□ V□□□□.

□□□□□□□□□□□□□□□□.

□□□□:□ F□□□□□□□ V□□□[①].

□ 8 □□ K□□□ $s\times n$□□□□□□□□□ $M_{s\times n}(K)$□,□□□□□ □□□□□□

$$\{E_{11}, E_{12}, \dots, E_{1n}, \dots, E_{s1}, E_{s2}, \dots, E_{sn}\}$$

□ $M_{s\times n}(K)$□□□□.

□□ □□ $s\times n$□□ $A=(a_{ij})$□□□□□

$$A=\sum_{i=1}^{s}\sum_{j=1}^{n} a_{ij}E_{ij}.$$

□□ $\sum_{i=1}^{s}\sum_{j=1}^{n} k_{ij}E_{ij}=0$,□□□$(k_{ij})$□□□□,□□

$$k_{ij}=0, 1\leq i\leq s, 1\leq j\leq n.$$

□□ $\{E_{11}, E_{12}, \dots, E_{1n}, \dots, E_{s1}, E_{s2}, \dots, E_{sn}\}$□□□□,□□□□ $M_{s\times n}(K)$□□ □□. x

□ 9 □□ K□□□□□□□□□□□□□□□ $K[x]$□,□□

$$S=\{1, x, x^2, \dots, x^n, \dots\}$$

□ $K[x]$□□□□.

□□ K□□□□□□□□□ $f(x)$□□□□

$$f(x)=a_0+a_1x+a_2x^2+\dots+a_nx^n.$$

□□ S□□□□□□□ $\{x^{i_1}, x^{i_2}, \dots, x^{i_m}\}$. □□

$$k_1x^{i_1}+k_2x^{i_2}+\dots+k_mx^{i_m}=0,$$

□□□□□□□□□□□, $k_1=k_2=\dots=k_m=0$,□□□□□□□□□□,□□ S□□□□,□□ S□ $K[x]$□□□□. x

□□ 8 V□□□□□□,□□ V□□□□□□□□□□□□;□□,V□ □□□□□.

① □□□□□□□□□□□□□□(□□)□(□□□□□□ 1996□ 12□□ 1□)□ 176□ 177 □.

□8□□ $M_{s\times n}(K)$□□□□□,□9□□ $K[x]$□□□□□(□□□□, $K[x]$□□□□□□□□□□□□□□□□).

□□ **10** □□ V□□□□□,□ V□□□□□□□□□□□□□□□.

□□ □□□8□, V□□□□□□□□□□□□:$\alpha_1,\alpha_2,\dots,\alpha_n$. □□□□ S□ V□□□□□. □□ S□□□□□□□□□ n□,□ S□□□□ $n+1$□□□:$\beta_1,\beta_2,\dots,\beta_{n+1}$. □□□□□ $\alpha_1,\alpha_2,\dots,\alpha_n$ □□□□. □□□ 1□, $\beta_1,\beta_2,\dots,\beta_{n+1}$□□□□. □□ S□□□□□□,□□ $|S|$□ n. □ $S=\{\beta_1,\beta_2,\dots,\beta_m\}$. □□□ 4□, $m=n$. x

□□ **9** □ V□□□□□,□ V□□□□□□□□□□□□□ V□□□,□□ $\dim_F V$,□□□ $\dim V$.

□□□□□□□□□□□□□□ 0.

□□ 8□□, $\dim M_{s\times n}(K)=sn$.

□□□□□□□□□□□□□□□□□□□□□□□.

□□ **11** □□ $\dim V=n$,□ V□□□ $n+1$□□□□□□□□.

□□ □□□10□□□□□□□□. x

□□ **12** □□ $\dim V=n$,□ V□□□ n□□□□□□□□□□ V□□□□.

□□ □ V□□□ n□□□□□□□□$\alpha_1,\alpha_2,\dots,\alpha_n$. □□□□ β□ V,□□□ 11□, $\alpha_1,\alpha_2,\dots,\alpha_n,\beta$□□□□. □□□□□ 2□, β□□□$\alpha_1,\alpha_2,\dots,\alpha_n$□□□□. □□ $\alpha_1,\alpha_2,\dots,\alpha_n$□ V□□□□. x

□□□□□□□□□□□□□□□□□□.

□□ **13** □ $\alpha_1,\alpha_2,\dots,\alpha_n$□ V□□□□,□ V□□□□□□α□□□□□□□$\alpha_1,\alpha_2,\dots,\alpha_n$□□□□□.

□□ □□□□□□□, α□□□$\alpha_1,\alpha_2,\dots,\alpha_n$□□□□. □□□□□□□□□:

$$\alpha=a_1\alpha_1+a_2\alpha_2+\dots+a_n\alpha_n,$$

$$\alpha=b_1\alpha_1+b_2\alpha_2+\dots+b_n\alpha_n,$$

□□

$$0=(a_1-b_1)\alpha_1+(a_2-b_2)\alpha_2+\dots+(a_n-b_n)\alpha_n.$$

□□ $\alpha_1,\alpha_2,\dots,\alpha_n$□□□□□,□□

$$a_1-b_1=0,\ a_2-b_2=0,\ \dots,\ a_n-b_n=0.$$

□□□□□□□□□□. x

□□□ α □□ $\alpha_1,\alpha_2,\dots,\alpha_n$□□□□□□□□□□ n□□□□($a_1,a_2,\dots,$

a_n)□□ α □□ $\alpha_1, \alpha_2, \ldots, \alpha_n$ □□□□. □□□□□ α □□□□□□□□□□□.

□□□□□,□□□□□□□ V □□□□□□,□ V □□□□□□□□□□□□□□□□□□□□□□□□□,□□ V □□□□□□□□.

n □□□□□ V □□□□□□,V □□□□□□□□□□□□□□□□□□□□□□□□?

□ $\alpha_1, \alpha_2, \ldots, \alpha_n$ □ $\beta_1, \beta_2, \ldots, \beta_n$ □ V □□□□. V □□□ α □□□□□□□□□□□□

$$X = (x_1, x_2, \ldots, x_n)□, \quad Y = (y_1, y_2, \ldots, y_n)□.$$

□□□ X □ Y □□□□□,□□□□□□□□□□□□□□□□. □□ $\alpha_1, \alpha_2, \ldots, \alpha_n$ □ V □□□□,□□□

$$\begin{aligned} \beta_1 &= a_{11}\alpha_1 + a_{21}\alpha_2 + \ldots + a_{n1}\alpha_n, \\ \beta_2 &= a_{12}\alpha_1 + a_{22}\alpha_2 + \ldots + a_{n2}\alpha_n, \\ &\ldots\ldots\ldots\ldots \\ \beta_n &= a_{1n}\alpha_1 + a_{2n}\alpha_2 + \ldots + a_{nn}\alpha_n. \end{aligned} \tag{2}$$

□□□□□□□□□,□□□□□□□□□□:

$$x_1\alpha_1 + x_2\alpha_2 + \ldots + x_n\alpha_n \stackrel{\text{def}}{=} (\alpha_1, \alpha_2, \ldots, \alpha_n) \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{pmatrix}. \tag{3}$$

□□□□□(2)□□□

$$(\beta_1, \beta_2, \ldots, \beta_n) = (\alpha_1, \alpha_2, \ldots, \alpha_n) \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}. \tag{4}$$

□□□(4)□□□□ n □□□□□ A,□□□□ $\alpha_1, \alpha_2, \ldots, \alpha_n$ □□ $\beta_1, \beta_2, \ldots, \beta_n$ □□□□□. □□(4)□□□□

$$(\beta_1, \beta_2, \ldots, \beta_n) = (\alpha_1, \alpha_2, \ldots, \alpha_n) A. \tag{5}$$

□□□□□□□□□□□□□□,□□□□□□□□□□□□□(□)□□□□□□□□□□□□□□□□□□□□,□□□□□□□□□□□□□□:

□ $\alpha_1, \alpha_2, \ldots, \alpha_n$ □ $\beta_1, \beta_2, \ldots, \beta_n$ □ V □□□□□□,A, B □□ F □□□ n □□□,k□ F,□

$$[(\alpha_1, \alpha_2, \ldots, \alpha_n) A] B = (\alpha_1, \alpha_2, \ldots, \alpha_n)(AB), \tag{6}$$

$$(\alpha_1,\alpha_2,\dots,\alpha_n)A+(\alpha_1,\alpha_2,\dots,\alpha_n)B=(\alpha_1,\alpha_2,\dots,\alpha_n)(A+B), \tag{7}$$

$$(\alpha_1,\alpha_2,\dots,\alpha_n)A+(\beta_1,\beta_2,\dots,\beta_n)A=(\alpha_1+\beta_1,\alpha_2+\beta_2,\dots,\alpha_n+\beta_n)A, \tag{8}$$

$$\begin{aligned}[k(\alpha_1,\alpha_2,\dots,\alpha_n)]A&=(\alpha_1,\alpha_2,\dots,\alpha_n)(kA)\\&=k[(\alpha_1,\alpha_2,\dots,\alpha_n)A],\end{aligned} \tag{9}$$

[illegible]

$$(\alpha_1,\alpha_2,\dots,\alpha_n)+(\beta_1,\beta_2,\dots,\beta_n)\overset{\text{def}}{=}(\alpha_1+\beta_1,\alpha_2+\beta_2,\dots,\alpha_n+\beta_n), \tag{10}$$

$$k(\alpha_1,\alpha_2,\dots,\alpha_n)\overset{\text{def}}{=}(k\alpha_1,k\alpha_2,\dots,k\alpha_n). \tag{11}$$

[illegible] **14** [illegible] $\alpha_1,\alpha_2,\dots,\alpha_n$ [illegible] V [illegible], [illegible]

$$(\beta_1,\beta_2,\dots,\beta_n)=(\alpha_1,\alpha_2,\dots,\alpha_n)A,$$

[illegible] $\beta_1,\beta_2,\dots,\beta_n$ [illegible] V [illegible] A [illegible].

[illegible] [illegible] $\alpha_1,\alpha_2,\dots,\alpha_n$ [illegible], [illegible]

$$k_1\beta_1+k_2\beta_2+\dots+k_n\beta_n=(\beta_1,\beta_2,\dots,\beta_n)\begin{pmatrix}k_1\\k_2\\\vdots\\k_n\end{pmatrix}=(\alpha_1,\alpha_2,\dots,\alpha_n)A\begin{pmatrix}k_1\\k_2\\\vdots\\k_n\end{pmatrix},$$

[illegible]

$\beta_1,\beta_2,\dots,\beta_n$ [illegible] V [illegible]

[illegible] $\beta_1,\beta_2,\dots,\beta_n$ [illegible]

[illegible] [illegible] $k_1\beta_1+k_2\beta_2+\dots+k_n\beta_n=0$ [illegible] $k_1=k_2=\dots=k_n=0$

[illegible] [illegible] $(\alpha_1,\alpha_2,\dots,\alpha_n)A\begin{pmatrix}k_1\\k_2\\\vdots\\k_n\end{pmatrix}=0$ [illegible] $\begin{pmatrix}k_1\\k_2\\\vdots\\k_n\end{pmatrix}=0$

[illegible] [illegible] $A\begin{pmatrix}k_1\\k_2\\\vdots\\k_n\end{pmatrix}=0$ [illegible] $\begin{pmatrix}k_1\\k_2\\\vdots\\k_n\end{pmatrix}=0$

[illegible] [illegible] $AZ=0$ [illegible]

$\Leftrightarrow$ $|A| \neq 0$

$\Leftrightarrow$ A□□□□□. $\times$

□□ 14 □□□□□□,□ $\alpha_1,\alpha_2,\ldots,\alpha_n$ □□ $\beta_1,\beta_2,\ldots,\beta_n$ □□□□□ A□□□□□.

□□□□□□□□ α □□□□ $\alpha_1,\alpha_2,\ldots,\alpha_n$ □□ $\beta_1,\beta_2,\ldots,\beta_n$ □□□□ X, Y □□□□□:□□

$$\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n)X,\ \alpha=(\beta_1,\beta_2,\ldots,\beta_n)Y,$$

□□□ $\alpha_1,\alpha_2,\ldots,\alpha_n$ □□ $\beta_1,\beta_2,\ldots,\beta_n$ □□□□□ A,□□

$$\begin{aligned}(\alpha_1,\alpha_2,\ldots,\alpha_n)X&=(\beta_1,\beta_2,\ldots,\beta_n)Y\\&=(\alpha_1,\alpha_2,\ldots,\alpha_n)AY.\end{aligned}$$

□□□□□□□□□ $\alpha_1,\alpha_2,\ldots,\alpha_n$ □□□□□□□□□,□□□□

$$X=AY, \tag{12}$$

□(12)□□

$$Y=A^{-1}X. \tag{13}$$

□□ 8.1

1. □□□□□□□□□□□□□□□□□□□□ R□□□□□□:

(1) R[x]□□□ 2□□□□□□□□□,□□□□□□□□□□□□□□;

(2) □□□□□□□□□□ R^+,□□□□□□□□□□□□

$$a \oplus b = ab,\ \forall\, a, b \in R^+,$$

$$k \circ a = a^k,\ \forall\, a \in R^+,\ k \in R;$$

(3) □□[a, b]□□□□□□□□□□□□□,□□ C[a, b],□□□□□□□□□□□.

2. □□□□□ R□□□□□□ R^R□□□□□□□□□□□□□□?

(1) $1, \cos^2 x, \cos 2x$;

(2) $1, \cos x, \cos 2x, \cos 3x$;

(3) $1, \sin x, \cos x$

(4) $\sin x, \cos x, \sin^2 x, \cos^2 x$;

(5) $1, e^x, e^{2x}, e^{3x}, \ldots, e^{nx}$;

(6) $x^2, x|x|$.

3. □□ 1□□(2)□□□□□□□□□□□□□□.

4. □□□□ C□□□□□ R□□□□□□,□□□□□□□□□,□□□□□□ $z = a+bi$ □□□□□□□□.

5. □□□ K□□□□□□□□□□,□□□□□□□□□.

6. □□□□ K□□□ n□□□□□□□□□□ V_1,□□□□□□□□□□□□□,□□ K□□□□□□□,□ V_1 □□□□□□□.

7. □□□□ K□□□ n□□□□□□□□□□□ V_2,□□□□□□□□□□□□□,□□ K□□□□□□□,□ V_2 □□□□□□□.

8. □□□□ K□□□ n□□□□□□□□□□□ W,□□□□□□□□□□□□□,□□ K□□□□□□□,□ W□□□□□□□.

9. □ K^3 □,□

$$\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},\ \alpha_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix},\ \alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix};$$

$$\beta_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},\ \beta_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix},\ \beta_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix},$$

□□ $\alpha_1,\alpha_2,\alpha_3$ □□ β_1,β_2,β_3 □□□□□,□□□□□ $\alpha=(2,5,3)'$□□□□□□□□□□□ X, Y.

10. □□:□□□ K□□ n□□□□□ V□,□□□□□□□□□□□□ $\alpha_1,\alpha_2,\dots,\alpha_n$ □□□□,□ $\alpha_1,\alpha_2,\dots,\alpha_n$ □ V□□□□.

11. □ $X=\{x_1,x_2,\dots,x_n\}$, F□□□□.□□□□ X□□□ F□□□□□□□□□□□ F^X,□□□ F□□□□□□□□□,□ F^X □□□□□□□,□□□□□ f□□□□□□□□.

12. □

$$V = \left\{ \begin{pmatrix} x_1 & x_2 + i x_3 \\ x_2 - i x_3 & -x_1 \end{pmatrix} \middle| x_1, x_2, x_3 \in R \right\}.$$

(1) □□ V□□□□□□□□□□□□□□□□□ R□□□□□□□□□;

(2) □ V□□□□□□□;

(3) □ V□□□ $\begin{pmatrix} x_1 & x_2 + i x_3 \\ x_2 - i x_3 & -x_1 \end{pmatrix}$ □□(2)□□□□□□□□□□□□□.

§ 2 □□□□□□□□,□□□□□

□□ K□ n□□□□□□□□□ $AX=0$□□□□ W□ K^n □□□□,□□□□□□□□□□□□ W□□□□□□□□□□□□□□□□□.

□□ K□□□ n□□□□□□□□□□ $M_n(K)$,□□□□□□□□□□□□□□ K□□□□□□□. K□□□ n□□□□□□□□□□ V_1,□□□□□□□□□□□□□□ K□□□□□□□(□□□ 8.1□□ 6□).□□ V_1 □ $M_n(K)$□□□,□□ V_1 □□□□□ $M_n(K)$□□□, V_1 □□□□□□

[illegible] $M_n(K)$ [illegible]. [illegible] V_1 [illegible] $M_n(K)$ [illegible].

[illegible], [illegible] [illegible].

[illegible] 1 [illegible] F [illegible] V [illegible] U [illegible] V [illegible] F [illegible], [illegible] U [illegible] V [illegible], [illegible].

[illegible], $\{0\}$ [illegible] V [illegible], [illegible] V [illegible], [illegible] 0.

[illegible], V [illegible] V [illegible]. 0 [illegible] V [illegible] V [illegible], [illegible] [illegible].

[illegible] 1 [illegible] F [illegible] V [illegible] U [illegible] V [illegible] U [illegible] V [illegible], [illegible]

1° u_1, u_2 [illegible] U [illegible] $u_1 + u_2$ [illegible] U;

2° u [illegible] U, k [illegible] F [illegible] ku [illegible] U.

[illegible] [illegible] 1 [illegible]. [illegible].

[illegible], V [illegible] U [illegible]. [illegible] V [illegible], [illegible] U [illegible] [illegible]; [illegible] 5°, 6°, 7°, 8° [illegible] 4 [illegible].

[illegible] U [illegible], [illegible] u [illegible] U. [illegible], $0u$ [illegible] U. [illegible] V [illegible], [illegible] $0u = 0$. [illegible] 0 [illegible] U, [illegible] V [illegible] U [illegible].

[illegible] α [illegible] U, [illegible], $(-1)\alpha$ [illegible] U. [illegible] V [illegible], [illegible] $(-1)\alpha = -\alpha$, [illegible] $-\alpha$ [illegible] U, [illegible] α [illegible] V [illegible] $-\alpha$ [illegible] α [illegible] U [illegible].

[illegible], U [illegible] F [illegible]. [illegible] U [illegible] V [illegible]. ×

[illegible] 1 [illegible] K [illegible] n [illegible] $K[x]_n$, [illegible] $K[x]_n$ [illegible] $K[x]$ [illegible].

[illegible] [illegible] $K[x]_n$ [illegible]. [illegible] n [illegible] n, [illegible] k [illegible] n [illegible] n, [illegible] $K[x]_n$ [illegible], [illegible] $K[x]_n$ [illegible] $K[x]$ [illegible]. ×

[illegible] 2 [illegible] U [illegible] F [illegible] n [illegible] V [illegible], [illegible]

$$\dim U \; [illegible] \; \dim V.$$

[illegible] [illegible] n [illegible] V [illegible] $n+1$ [illegible], [illegible] U [illegible] n, [illegible]

$$\dim U \; [illegible] \; \dim V. \qquad ×$$

[illegible] 3 [illegible] U [illegible] F [illegible] n [illegible] V [illegible], [illegible] $\dim U = \dim V$, [illegible] $U = V$.

[illegible] [illegible] $\dim U = \dim V = n$, [illegible] U [illegible] $\delta_1, \delta_2, \ldots, \delta_n$ [illegible] V [illegible]

□□，□□ V□□□□□ $\alpha = a_1\delta_1 + a_2\delta_2 + \dots + a_n\delta_n$□ U，□□ $V \subseteq U$.□□ $U \subseteq V$，□□ $U = V$. x

□□ 4 □ U□□ F□ n□□□□□ V□□□□□□，□ U□□□□□□□□□ V□□□□.

□□ □ $\alpha_1, \alpha_2, \dots, \alpha_s$□ U□□□□，□ s□ n.□□ $s = n$，□ $\alpha_1, \alpha_2, \dots, \alpha_n$□ V□□□□.□□□ $s < n$.□□ $\alpha_1, \alpha_2, \dots, \alpha_s$□□ V□□□□，□□ V□□□□□□□□ β_1□□□ $\alpha_1, \alpha_2, \dots, \alpha_s$□□□□，□□ $\alpha_1, \alpha_2, \dots, \alpha_s, \beta_1$□□□□.□□ $s+1 = n$，□□□□ V□□□□.□□ $s+1 < n$，□□□□ β_2□ V，□□ $\alpha_1, \alpha_2, \dots, \alpha_s, \beta_1, \beta_2$□□□□.□□□□，□□□□，□□ n□□□□□□□□ $\alpha_1, \alpha_2, \dots, \alpha_s, \beta_1, \beta_2, \dots, \beta_r$，□□ $s + r = n$，□□□ V□□□□. x

□□□□□ F□□□□□ V□□□□?□□□□□ 3□§1□□□□□□□□，V□□□□□□ $\alpha_1, \alpha_2, \dots, \alpha_s$，□□□□□□□□□□□□□□□

$$\{k_1\alpha_1 + k_2\alpha_2 + \dots + k_s\alpha_s \mid k_1, k_2, \dots, k_s \text{□} F\}$$

□ V□□□□□□，□□□□ $\alpha_1, \alpha_2, \dots, \alpha_s$□□□□□□，□□□ $\alpha_1, \alpha_2, \dots, \alpha_s$□.

□□ 3□§4□□□□□，□□□□□□□□:

□□ 5 □□ F□□□□□□ V□，□□

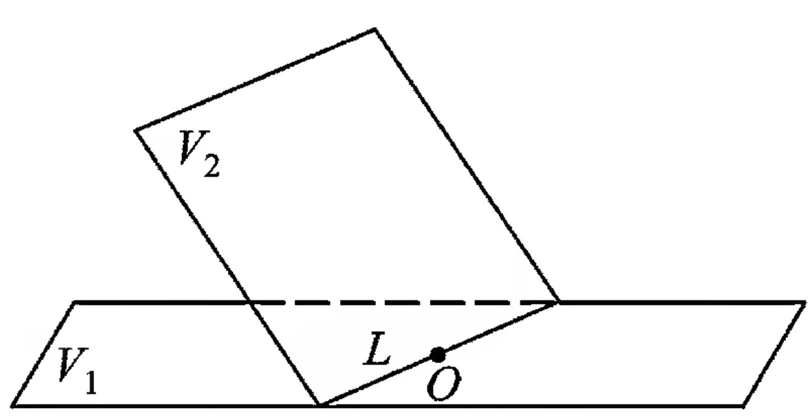
$$U = \text{□}\alpha_1, \alpha_2, \dots, \alpha_s\text{□},$$

□□□□ $\alpha_1, \alpha_2, \dots, \alpha_s$□□□□□□□□□□□ U□□□□，□□

$$\dim U = \operatorname{rank}\{\alpha_1, \alpha_2, \dots, \alpha_s\}.$$ x

□□□□□□□□□□，□□ $\delta_1, \delta_2, \dots, \delta_r$□ V□□□□ U□□□□，□ $U = \text{□}\delta_1, \delta_2, \dots, \delta_r\text{□}$.□□□□，$V$□□□□□□□□□□□□□□□□□□□□.

□□□□ V□，□□□□□□□ O□□□ V_1，V_2，□□□ V□□□□，□□ 8-1□□.□ V_1□ V_2□□□ V□□□□□□□?



□ 8-1

□□□□□□□□□□，□□ V_1□ V_2□□□ L□ V□□□□□□.□□□，□□□□□□:

□□ 6 □ V_1, V_2 □□□ F □□□□□ V □□□□,□ V_1 □ V_2 □□ V □□□□.

□□ □□ 0□ V_1 □ V_2,□□ V_1 □ V_2 □□□.□ α, β □ V_1 □ V_2,□ α, β □ V_i, $i = 1, 2$. □□ $\alpha + \beta$ □ V_i, $i = 1, 2$, □□ $\alpha + \beta$ □ V_1 □ V_2. □□□□, V_1 □ V_2 □□ V □□□□□□□.□□ V_1 □ V_2 □ V □□□□. X

□□□□□□□□□□□□□,□

$$V_1 □ V_2 = V_2 □ V_1, (V_1 □ V_2) □ V_3 = V_1 □ (V_2 □ V_3).$$

□□□□,□□□□□□□□□□□□:$\square_{i=1}^{s} V_i$,□□ V □□□□□.

V_1 □ V_2 □□□ V □□□□□□?□□□□□□□□□□□□□,□□ α_i □ V_i,□ α_i | L, $i = 1, 2$, □□□ α_i □ V_1 □ V_2, $i = 1, 2$, □□ $\alpha_1 + \alpha_2$ □□□□ □ V_1 □ V_2,□□ V_1 □ V_2 □□ V □□□□.□□□□□□□□□□□ V_1 □ V_2 □□□□,□□□□□□□□□□□ V_1 □□□□□ α_1 □ V_2 □□□□□ α_2 □□.□□□□□□,□□□□□□□□□□

$$\{\alpha_1 + \alpha_2 \mid \alpha_1 □ V_1, \alpha_2 □ V_2\}.$$

□□ 7 □ V_1, V_2 □□□ F □□□□□ V □□□□,□ V □□□

$$\{\alpha_1 + \alpha_2 \mid \alpha_1 □ V_1, \alpha_2 □ V_2\}$$

□ V □□□□□□,□□□ V_1 □ V_2 □□,□□ $V_1 + V_2$,□

$$V_1 + V_2 = \{\alpha_1 + \alpha_2 \mid \alpha_1 □ V_1, \alpha_2 □ V_2\}. \tag{1}$$

□□ □□ $0 = 0 + 0$, □□ 0□ $V_1 + V_2$. □ $V_1 + V_2$ □□□□□□□ α, β, □ $\alpha = \alpha_1 + \alpha_2$, $\beta = \beta_1 + \beta_2$, □□ α_1, β_1 □ V_1, α_2, β_2 □ V_2. □□ $\alpha_1 + \beta_1$ □ V_1, $\alpha_2 + \beta_2$ □ V_2. □□

$$\begin{aligned} \alpha + \beta &= (\alpha_1 + \alpha_2) + (\beta_1 + \beta_2) \\ &= (\alpha_1 + \beta_1) + (\alpha_2 + \beta_2) □ V_1 + V_2. \end{aligned}$$

□□□□ $V_1 + V_2$ □□ V □□□□□□□,□□ $V_1 + V_2$ □ V □□□□□□. X

□ U □ V □□□□,□ U� V_1 □ V_2,□ U� $V_1 + V_2$. □□□ $V_1 + V_2$ □ V □□□ V_1 □ V_2 □□□□□□.

□(1)□□□□□,□□□□□□□□□□□□□:

1° $V_1 + V_2 = V_2 + V_1$ (□□□);

2° $(V_1 + V_2) + V_3 = V_1 + (V_2 + V_3)$ (□□□).

□□□□,□□□□□□□□□□□□□:

$$V_1 + V_2 + \dots + V_s$$

$$=\{\alpha_1+\alpha_2+\dots+\alpha_s \mid \alpha_i \in V_i,\ i=1,2,\dots,s\}, \quad (2)$$

□□□ V□□□□□□.

□□ **8** □ $\alpha_1,\alpha_2,\dots,\alpha_s$□ $\beta_1,\beta_2,\dots,\beta_r$□□ F□□□□□ V□□□□□□,□

$$\langle\alpha_1,\alpha_2,\dots,\alpha_s\rangle+\langle\beta_1,\beta_2,\dots,\beta_r\rangle$$
$$=\langle\alpha_1,\alpha_2,\dots,\alpha_s,\beta_1,\beta_2,\dots,\beta_r\rangle.$$

□□ □□□□□□□□□□□□□□□□□□□□□□□□□,□□

$$\langle\alpha_1,\alpha_2,\dots,\alpha_s\rangle+\langle\beta_1,\beta_2,\dots,\beta_r\rangle$$
$$=\{(k_1\alpha_1+k_2\alpha_2+\dots+k_s\alpha_s)+(l_1\beta_1+l_2\beta_2+\dots+l_r\beta_r) \mid$$
$$k_i, l_j \in F, 1\le i\le s, 1\le j\le r\}$$
$$=\langle\alpha_1,\alpha_2,\dots,\alpha_s,\beta_1,\beta_2,\dots,\beta_r\rangle. \qquad \times$$

□□ 8-1 □□, $\dim V_1=\dim V_2=2$, $\dim(V_1\cap V_2)=1$, □□□□, $V_1+V_2=V$, □□ $\dim(V_1+V_2)=3$, □□□

$$\dim V_1+\dim V_2=\dim(V_1+V_2)+\dim(V_1\cap V_2).$$

□□□□ n□□□□□,□□□□□□?□□□□□.

□□ **9(**□□□□□□□□**)** □ V_1, V_2□□□ F□□□□□ V□□□□□□□,□ $V_1\cap V_2$, V_1+V_2□□□□□□□□□,□□

$$\dim V_1+\dim V_2=\dim(V_1+V_2)+\dim(V_1\cap V_2). \quad (3)$$

□□ □□ $V_1\cap V_2\subseteq V_1$, □□ $\dim(V_1\cap V_2)\le\dim V_1$. □ V_1, V_2, $V_1\cap V_2$□□□□□ n_1, n_2, m□ $V_1\cap V_2$□□□□ $\alpha_1,\alpha_2,\dots,\alpha_m$, □□□□□□□ V_1, V_2□□□□:

$$\alpha_1,\alpha_2,\dots,\alpha_m,\beta_1,\dots,\beta_{n_1-m},$$
$$\alpha_1,\alpha_2,\dots,\alpha_m,\gamma_1,\dots,\gamma_{n_2-m},$$

□□

$$V_1+V_2$$
$$=\langle\alpha_1,\alpha_2,\dots,\alpha_m,\beta_1,\dots,\beta_{n_1-m}\rangle+\langle\alpha_1,\alpha_2,\dots,\alpha_m,\gamma_1,\dots,\gamma_{n_2-m}\rangle$$
$$=\langle\alpha_1,\alpha_2,\dots,\alpha_m,\beta_1,\dots,\beta_{n_1-m},\gamma_1,\dots,\gamma_{n_2-m}\rangle.$$

□□□□□□□ $\alpha_1,\alpha_2,\dots,\alpha_m,\beta_1,\dots,\beta_{n_1-m},\gamma_1,\dots,\gamma_{n_2-m}$□□□□,□□□ V_1+V_2□□□□,□□□□

$$\dim(V_1+V_2)=m+(n_1-m)+(n_2-m)=n_1+n_2-m$$
$$=\dim V_1+\dim V_2-\dim(V_1\cap V_2).$$

□□□□□

$$k_1\alpha_1+k_2\alpha_2+\cdots+k_m\alpha_m+p_1\beta_1+\cdots+p_{n_1-m}\beta_{n_1-m}+q_1\gamma_1+\cdots+q_{n_2-m}\gamma_{n_2-m}=0, \tag{4}$$

□

$$q_1\gamma_1+\cdots+q_{n_2-m}\gamma_{n_2-m}=-k_1\alpha_1-\cdots-k_m\alpha_m-p_1\beta_1-\cdots-p_{n_1-m}\beta_{n_1-m}. \tag{5}$$

(5)□□□□□□□□ V_2,□□□□□□□ V_1,□□□□□□□□□ $V_1\cap V_2$,□□□□□ $\alpha_1,\cdots,\alpha_m$ □□□□:

$$q_1\gamma_1+\cdots+q_{n_2-m}\gamma_{n_2-m}=l_1\alpha_1+\cdots+l_m\alpha_m.$$

□□□

$$l_1\alpha_1+l_2\alpha_2+\cdots+l_m\alpha_m-q_1\gamma_1-\cdots-q_{n_2-m}\gamma_{n_2-m}=0. \tag{6}$$

□□ $\alpha_1,\alpha_2,\cdots,\alpha_m,\gamma_1,\cdots,\gamma_{n_2-m}$ □ V_2 □□□□,□□□(6)□□□

$$l_1=l_2=\cdots=l_m=q_1=\cdots=q_{n_2-m}=0,$$

□□(4)□,□

$$k_1\alpha_1+k_2\alpha_2+\cdots+k_m\alpha_m+p_1\beta_1+\cdots+p_{n_1-m}\beta_{n_1-m}=0.$$

□□□□

$$k_1=k_2=\cdots=k_m=p_1=\cdots=p_{n_1-m}=0.$$

□□ $\alpha_1,\alpha_2,\cdots,\alpha_m,\beta_1,\cdots,\beta_{n_1-m},\gamma_1,\cdots,\gamma_{n_2-m}$ □□□□. ×

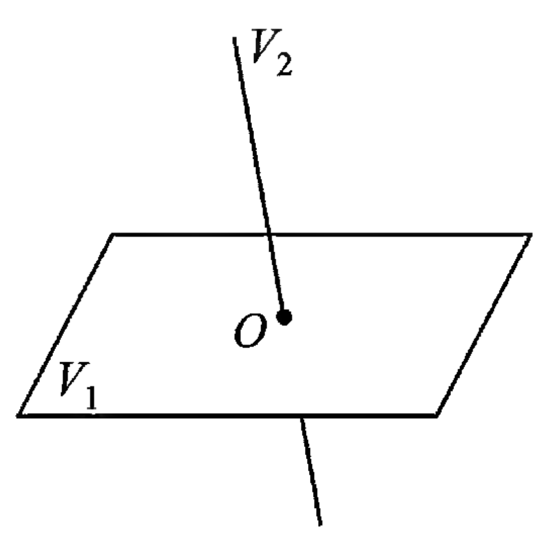
□□ 10 □ V_1,V_2 □□□ F □ n □□□□□ V □□□□,□

$$\dim(V_1+V_2)=\dim V_1+\dim V_2 \Leftrightarrow V_1\cap V_2=0.$$ ×

□□□□ V □,V_1 □□□□□□□□□,V_2 □□□□□□□□□,□ V_2 □□□□ V_1 □.□□ 8-2 □□,□□□□,$V_1+V_2=V$,□ V_1+V_2 □□□□□ α □□□□□□□□

$$\alpha=\alpha_1+\alpha_2,\ \alpha_1\in V_1,\ \alpha_2\in V_2.$$

□□□□□□,□□□□□□:



□ 8-2

□□ 2 □ V_1, V_2□□ F□□□□□ V□□□□,□□ V_1+V_2□□□□□α□□□□□□□

$$\alpha=\alpha_1+\alpha_2,\ \alpha_1\in V_1,\ \alpha_2\in V_2, \tag{7}$$

□□ V_1+V_2□□□,□□ $V_1\oplus V_2$.

□□ 11 □ V_1, V_2□□ F□□□□□ V□□□□□□□,□□□□□□□□:

(i) V_1+V_2□□□;

(ii) V_1+V_2□□□□□□□□□□;

(iii) $V_1\cap V_2=0$;

(iv) $\dim(V_1+V_2)=\dim V_1+\dim V_2$;

(v) V_1□□□□□ V_2□□□□□□□ V_1+V_2□□□□.

□□ (i)$\Leftrightarrow$(ii)□□□□.

(ii)$\Rightarrow$(iii). □□ $\alpha\in V_1\cap V_2$,□□□□□□□□

$$0=\alpha+(-\alpha),\ \alpha\in V_1,\ -\alpha\in V_2.$$

□□□□□□,$\alpha=0$. □□ $V_1\cap V_2=0$.

(iii)$\Rightarrow$(i). □□ $\alpha\in V_1+V_2$,□□ α□□□□□:

$$\alpha=\alpha_1+\alpha_2,\quad \alpha_1\in V_1,\ \alpha_2\in V_2,$$

$$\alpha=\beta_1+\beta_2,\quad \beta_1\in V_1,\ \beta_2\in V_2,$$

□ $\alpha_1+\alpha_2=\beta_1+\beta_2$,□□□□ $\alpha_1-\beta_1=\beta_2-\alpha_2\in V_1\cap V_2$. □□ $V_1\cap V_2=0$,□□ $\alpha_1=\beta_1, \alpha_2=\beta_2$,□□ V_1+V_2□□□.

(iii)$\Leftrightarrow$(iv). □□□□ 10□□□□.

(iv)$\Rightarrow$(v). □ $\alpha_1,\alpha_2,\dots,\alpha_s$□ V_1□□□□,$\beta_1,\beta_2,\dots,\beta_r$□ V_2□□□□,□

$$\begin{aligned}V_1+V_2&=\langle\alpha_1,\alpha_2,\dots,\alpha_s\rangle+\langle\beta_1,\beta_2,\dots,\beta_r\rangle\\&=\langle\alpha_1,\alpha_2,\dots,\alpha_s,\beta_1,\beta_2,\dots,\beta_r\rangle.\end{aligned}$$

□□ $\dim(V_1+V_2)=\dim V_1+\dim V_2=s+r$,□ V_1+V_2□□□□□□□□ $\alpha_1,\alpha_2,\dots,\alpha_s,\beta_1,\beta_2,\dots,\beta_s$□□□□,□□ $\alpha_1,\alpha_2,\dots,\alpha_s,\beta_1,\beta_2,\dots,\beta_r$□ V_1+V_2□□□□(□□□□ 8.1□ 10□).

(v)$\Rightarrow$(iv) □□□□. X

□ V_1, V_2□□□□□□ V□□□□,□□□□□□:

1° $V_1+V_2=V$;

2° V_1+V_2□□□,

□□ V□ V_1□ V_2□□□,□□ $V=V_1\oplus V_2$. □□□ V_1□ V_2□□□□□□,□□ V_2□ V_1□□□□□□.

[illegible] 12 [illegible] V [illegible] F [illegible] n [illegible],[illegible] V [illegible] U [illegible].

[illegible] U [illegible] $\alpha_1, \alpha_2, \ldots, \alpha_m$, [illegible] V [illegible]: $\alpha_1, \alpha_2, \ldots, \alpha_m, \alpha_{m+1}, \ldots, \alpha_n$, [illegible]

$$V = \langle \alpha_1, \alpha_2, \ldots, \alpha_m, \alpha_{m+1}, \ldots, \alpha_n \rangle$$
$$= \langle \alpha_1, \alpha_2, \ldots, \alpha_m \rangle + \langle \alpha_{m+1}, \ldots, \alpha_n \rangle = U + W,$$

[illegible] $W = \langle \alpha_{m+1}, \ldots, \alpha_n \rangle$. [illegible] U [illegible] W [illegible] V [illegible],[illegible] $U + W$ [illegible],[illegible]

$$V = U \oplus W.$$

[illegible] W [illegible] U [illegible]. ∎

[illegible] 2 [illegible] $V = M_n(K)$, [illegible] K [illegible]. [illegible] V_1, V_2 [illegible] K [illegible] n [illegible],[illegible]:

$$V = V_1 \oplus V_2.$$

[illegible],[illegible] $V_1 + V_2 = V$. [illegible] $V_1 + V_2 \subseteq V$. [illegible] $V \subseteq V_1 + V_2$. [illegible] $A \in V = M_n(K)$, [illegible]

$$A = \frac{A + A'}{2} + \frac{A - A'}{2}. \tag{8}$$

[illegible] $\frac{A + A'}{2}$ [illegible], $\frac{A - A'}{2}$ [illegible],[illegible] (8) [illegible], $A \in V_1 + V_2$. [illegible] $V \subseteq V_1 + V_2$. [illegible] $V = V_1 + V_2$.

[illegible],[illegible] $V_1 + V_2$ [illegible],[illegible] $V_1 \cap V_2 = 0$. [illegible] $B \in V_1 \cap V_2$, [illegible] $B' = B$, [illegible] $B' = -B$, [illegible] $B = -B$, [illegible] $2B = 0$, [illegible] $B = 0$. [illegible] $V_1 \cap V_2 = 0$.

[illegible], $V = V_1 \oplus V_2$. ∎

[illegible].

[illegible] 3 [illegible] $V_1, V_2, \ldots, V_s$ [illegible] F [illegible] V [illegible],[illegible] $V_1 + V_2 + \ldots + V_s$ [illegible] α [illegible]

$$\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_s, \ \alpha_i \in V_i, \ i = 1, 2, \ldots, s, \tag{9}$$

[illegible] $V_1 + V_2 + \ldots + V_s$ [illegible],[illegible] $V_1 \oplus V_2 \oplus \ldots \oplus V_s$ [illegible] $\bigoplus_{i=1}^{s} V_i$.

[illegible]:

[illegible] 13 [illegible] $V_1, V_2, \ldots, V_s$ [illegible] F [illegible] V [illegible],[illegible]:

(i) $V_1 + V_2 + \ldots + V_s$ [illegible];

(ii) $V_1 + V_2 + \ldots + V_s$ [illegible];

(iii) $V_i \cap \sum_{j \neq i} V_j = 0,\ i = 1, 2, \dots, s$;

(iv) $\dim(V_1 + V_2 + \dots + V_s) = \dim V_1 + \dim V_2 + \dots + \dim V_s$;

(v) $V_i\ (i = 1, 2, \dots, s)$□□□□,□□□□ $V_1 + V_2 + \dots + V_s$□□□□.

□□□□□□:

$$\begin{array}{c} (\mathrm{i}) \Leftrightarrow (\mathrm{ii}) \Leftrightarrow (\mathrm{iii}) \Leftrightarrow (\mathrm{iv}) \Leftrightarrow (\mathrm{v}) \\ \qquad\qquad \Updownarrow \qquad\qquad\qquad \Updownarrow \\ \qquad\qquad (\mathrm{i}) \qquad\qquad\qquad (\mathrm{ii}) \end{array}$$

□ **3** □ $V = K^4$, $V_1 = \langle \alpha_1, \alpha_2, \alpha_3 \rangle$, $V_2 = \langle \beta_1, \beta_2 \rangle$,□□

$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 0 \\ 3 \\ 2 \\ 1 \end{pmatrix}, \beta_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \beta_2 = \begin{pmatrix} 1 \\ -1 \\ 3 \\ 7 \end{pmatrix},$$

□□□ $V_1 + V_2$, $V_1 \cap V_2$□□□□□□.

□ □□

$V_1 + V_2 = \langle \alpha_1, \alpha_2, \alpha_3 \rangle + \langle \beta_1, \beta_2 \rangle = \langle \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2 \rangle$,□□□□□ $\alpha_1, \alpha_2, \alpha_3$, β_1, β_2□□□□□□□□□□□□ $V_1 + V_2$□□□□,□□□□□□□□□ $V_1 + V_2$□□□,□□□ $A = (\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2)$,□ A□□□□□□□□□,□ □□□□□□□□□:

$$A = \begin{pmatrix} 1 & -1 & 0 & 2 & 1 \\ 2 & 1 & 3 & -1 & -1 \\ 1 & 1 & 2 & 0 & 3 \\ 0 & 1 & 1 & 1 & 7 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{10}$$

□□□□,$\alpha_1, \alpha_2, \beta_1$□ $V_1 + V_2$□□□□,$\dim(V_1 + V_2) = 3$.□□□□□,β_2□ □ $\alpha_1, \alpha_2, \beta_1$□□□□,□□□□□□□□□□□

$$x_1 \alpha_1 + x_2 \alpha_2 + x_3 \beta_1 = \beta_2$$

□□.□□□□ A□□□□□□□□□□□□ 1, 2, 4, 5□□□□□,□□□□ □□□$(-1, 4, 3)$.□□

$$\beta_2 = -\alpha_1 + 4\alpha_2 + 3\beta_1,$$

□□ $$\alpha_1 - 4\alpha_2 = 3\beta_1 - \beta_2 \in V_1 \cap V_2.$$

□□ $$\dim(V_1 \cap V_2) = \dim V_1 + \dim V_2 - \dim(V_1 + V_2),$$

□□(10)□□□□,$\dim V_1 = 2$, $\dim V_2 = 2$,□□

$$\dim(V_1 \cap V_2) = 2 + 2 - 3 = 1,$$

□□ $\alpha_1 - 4\alpha_2 = (5, -2, -3, -4)$□□ $V_1 \cap V_2$□□□□.

□ 3□□□□□□,□□□□□ A□□□□□□□□□□□□□□□□

□□□□,□□□□□□□□□□□□□□□□□□.

□ □ 8.2

1. □□□□ K□□□ n□□□□□□□□□ K^n□□□□:

(1) $a_1x_1+a_2x_2+\dots+a_nx_n=0$;

(2) $a_1x_1+a_2x_2+\dots+a_nx_n=1$;

(3) $x_1^2+x_2^2+x_3^2+\dots+x_{n-1}^2-x_n^2=0$.

2. □ A□□□ K□□□□ n□□□.□□:□□ K□□□□ A□□□□□□□□□□□□ $M_n(K)$□□□□□□,□□□□ $C(A)$.

3. □ $A=\operatorname{diag}\{a_1,a_2,\dots,a_n\}$,□□ $a_1,a_2,\dots,a_n$□□□ K□□□□□□□,□ $C(A)$□□□□□□.

4. □□□ K□□ 3□□□

$$A=\begin{pmatrix}0&0&1\\1&0&0\\4&-2&1\end{pmatrix},$$

□ $C(A)$□□□□□□□.

5. □□ F□□□□□□ V□,□□

$$k_1\alpha+k_2\beta+k_3\gamma=0, \text{□} \quad k_1k_2\text{□}0,$$

□□:□α,γ□=□β,γ□.

6. □ K^4□(K□□□),□□□□ $\alpha_1,\alpha_2,\alpha_3,\alpha_4$□□□□□□□□□□□□□.□

$$\alpha_1=(1,-3,2,-1)\text{□},\qquad \alpha_2=(-2,1,5,3)\text{□},$$

$$\alpha_3=(4,-3,7,1)\text{□},\qquad \alpha_4=(-1,-11,8,-3)\text{□}.$$

7. □ V□□ F□□□ n□□□□□,$\alpha_1,\alpha_2,\dots,\alpha_n$□ V□□□□,$\beta_1,\beta_2,\dots,\beta_s$□ V□□□□□,□□

$$(\beta_1,\beta_2,\dots,\beta_s)=(\alpha_1,\alpha_2,\dots,\alpha_n)A.$$

□□:□$\beta_1,\beta_2,\dots,\beta_s$□□□□□□ $n\times s$□□ A□□.

*8. □□□□□

$$f(x_1,x_2,\dots,x_n)=x_1^2+\dots+x_p^2-x_{p+1}^2-\dots-x_n^2$$

□□□□ $s>0$,□□:□□ $f(x_1,\dots,x_n)=0$□□□ W□□□□□□ W_1□ R^n□□□□□□□,□ $\dim W_1=n-p$.

9. □ V_1,V_2□□ F□□□□□ V□□□□□□□(□ V_i□ V),□□:

$$V_1\text{□}V_2\text{□}V.$$

*10. □ $V_1,V_2,\dots,V_s$□□ F□□□□□ V□ s□□□□□,□□:□□ $\operatorname{char}F=0$,□ V□□□□□□□□□□□ $V_1,V_2,\dots,V_s$□□□□.

11. □ $V=K^4$, $V_1=$□α_1,α_2□, $V_2=$□β_1,β_2□,□□

$$\alpha_1=\begin{pmatrix}1\\-1\\0\\1\end{pmatrix},\alpha_2=\begin{pmatrix}-2\\3\\1\\-3\end{pmatrix},\beta_1=\begin{pmatrix}1\\2\\0\\-2\end{pmatrix},\beta_2=\begin{pmatrix}1\\3\\1\\-3\end{pmatrix},$$

□□□ V_1+V_2, V_1□ V_2□□□□□□□.

12. □ $V=K^4$, $V_1=$□$\alpha_1,\alpha_2,\alpha_3$□, $V_2=$□β_1,β_2□, □□

$$\alpha_1=\begin{pmatrix}1\\1\\-1\\2\end{pmatrix},\alpha_2=\begin{pmatrix}2\\-1\\3\\0\end{pmatrix},\alpha_3=\begin{pmatrix}0\\-3\\5\\-4\end{pmatrix},\beta_1=\begin{pmatrix}1\\2\\2\\1\end{pmatrix},\beta_2=\begin{pmatrix}4\\-3\\3\\1\end{pmatrix},$$

□□□ V_1+V_2, V_1□ V_2□□□□□□□.

13. □ $V=K^n$, K□□□,□□□□□□□

$$x_1+x_2+\ldots+x_n=0$$

□□□□□ V_1;□□□□□□□

$$\begin{aligned}x_1-x_2&=0,\\x_1-x_3&=0,\\&\cdots\cdots\\x_1-x_n&=0\end{aligned}$$

□□□□□ V_2.□□: $V=V_1\oplus V_2$.

14. □□:□□ K□□□□ n□□□□□ V□□□□□□ n□□□□□□□□□.

15. □ $M_n^0(K)$□□ $M_n(K)$□□□□□□□□□□□□, K□□□.

(1) □□: $M_n^0(K)$□□□□□ $M_n(K)$□□□□□□;

(2) □□: $M_n(K)=$□I□$\oplus M_n^0(K)$.

*16. □ A□□□ K□□ n□□□,$\lambda_1,\lambda_2,\ldots,\lambda_s$□ A□□□□□□□□□,□ V_{λ_j}□□ A□□□λ_j□□□□□□.□□: A□□□□□□□□□□□□□□□

$$K^n=V_{\lambda_1}\oplus V_{\lambda_2}\oplus\ldots\oplus V_{\lambda_s}.$$

17. □ K□□□, $f_1(x)$, $f_2(x)$□ $K[x]$, A□ $M_n(K)$.□ $f(x)=f_1(x)f_2(x)$.□ W, W_1, W_2□□□□□□□□□□□

$$f(A)X=0,\quad f_1(A)X=0,\quad f_2(A)X=0$$

□□□□.

(1) □□: W_1, W_2□□ W□□□□;

(2) □□:□□$(f_1(x),f_2(x))=1$,□ $W=W_1\oplus W_2$.

§3 □□□□□□

□ F□ n□□□□□ V□□ F□ n□□□□□□□□□□□ F^n□□□

□.□□,□□ F^n □□□□ $\alpha_1,\alpha_2,\ldots,\alpha_s$ □□□□□ $U=$□$\alpha_1,\alpha_2,\ldots,\alpha_s$□,□□□ $\alpha_1,\alpha_2,\ldots,\alpha_s$ □□□□□□□□□□□ U □□□□, $\dim U$ □□ $\operatorname{rank}\{\alpha_1,\alpha_2,\ldots,\alpha_s\}$. □□ V □□□□□□□□□□□□□□□□□.

□□□□ F □ n □□□□□ V □ F^n □□□□? □□□□□□□□□□□□□□□□.

□□ **1** □ V □ V□□□ F □□□□□□,□□□ V □ V□□□□□□ σ,□□□□□□ α,β□ V, k□ F, □

$$\sigma(\alpha+\beta)=\sigma(\alpha)+\sigma(\beta), \tag{1}$$

$$\sigma(k\alpha)=k\sigma(\alpha), \tag{2}$$

□□□ σ □ V □ V□□□□□□□(□□□□□). □□ V □ V□□□□□□□,□□ V □ V□□□□,□□

$$V \cong V□.$$

□ F □□□□□ V □ V□□□□□□□ σ □□□□□□.

□□ **1** $\sigma(0)$□ V□□□□ 0□.

□□ □□ $0\alpha=0$,□□

$$\sigma(0)=\sigma(0\alpha)=0\sigma(\alpha)=0□.$$ x

□□ **2** □□□□ α□ V, □ $\sigma(-\alpha)=-\sigma(\alpha)$.

□□ $\sigma(-\alpha)=\sigma((-1)\alpha)=(-1)\sigma(\alpha)=-\sigma(\alpha)$. x

□□ **3** □□ V □□□□□ $\alpha_1,\alpha_2,\ldots,\alpha_s$, F □□□□□□ $k_1,k_2,\ldots,k_s$, □

$$\sigma(k_1\alpha_1+k_2\alpha_2+\ldots+k_s\alpha_s)=k_1\sigma(\alpha_1)+k_2\sigma(\alpha_2)+\ldots+k_s\sigma(\alpha_s).$$

□□ □□□ 1□□. x

□□ **4** V □□□□ $\alpha_1,\alpha_2,\ldots,\alpha_s$ □□□□□□□□ $\sigma(\alpha_1),\sigma(\alpha_2),\ldots,\sigma(\alpha_s)$□ V□□□□□□□□□.

□□ □□ σ □ V □ V□□□□□,□□□□ $\sigma(\alpha)=\sigma(\beta)$, □ $\alpha=\beta$. □□□

$$k_1\alpha_1+k_2\alpha_2+\ldots+k_s\alpha_s=0$$
$$\Rightarrow \sigma(k_1\alpha_1+k_2\alpha_2+\ldots+k_s\alpha_s)=\sigma(0)$$
$$\Rightarrow k_1\sigma(\alpha_1)+k_2\sigma(\alpha_2)+\ldots+k_s\sigma(\alpha_s)=0□,$$

□□ $\alpha_1,\alpha_2,\ldots,\alpha_s$ □□□□□□□ $\sigma(\alpha_1),\sigma(\alpha_2),\ldots,\sigma(\alpha_s)$ □□□□. x

□□ **5** □□ $\alpha_1,\alpha_2,\ldots,\alpha_n$ □ V □□□□, □ $\sigma(\alpha_1),\sigma(\alpha_2),\ldots,\sigma(\alpha_n)$ □ V□□□□. x

□□ □□□ 4□, $\sigma(\alpha_1),\sigma(\alpha_2),\ldots,\sigma(\alpha_n)$ □ V□□□□□□□□□□. □□ β□ V□, □□ σ □ V □ V□□□□□, □□□□ α□ V, □□ $\sigma(\alpha)=\beta$.

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n,$$

$$\beta = \sigma(\alpha) = a_1\sigma(\alpha_1) + a_2\sigma(\alpha_2) + \dots + a_n\sigma(\alpha_n),$$

$\sigma(\alpha_1), \sigma(\alpha_2), \dots, \sigma(\alpha_n)$ V' . ×

1 F .

V V' F .

5 .

. $\dim V = \dim V' = n$. V, V' : $\alpha_1, \alpha_2, \dots, \alpha_n$; $\gamma_1, \gamma_2, \dots, \gamma_n$.

$$\sigma: V \to V'$$

$$\alpha = \sum_{i=1}^{n} a_i\alpha_i \longmapsto \sum_{i=1}^{n} a_i\gamma_i. \qquad (3)$$

(3) , V α V' α . $\gamma_1, \gamma_2, \dots, \gamma_n$ V' , V V' ; V' $\delta = \sum_{i=1}^{n} b_i\gamma_i$, V $\beta = \sum_{i=1}^{n} b_i\alpha_i$ δ. σ V V' . $\alpha = \sum_{i=1}^{n} a_i\alpha_i$, $\beta = \sum_{i=1}^{n} b_i\alpha_i$, k F,

$$\sigma(\alpha+\beta) = \sigma\left(\sum_{i=1}^{n}(a_i+b_i)\alpha_i\right) = \sum_{i=1}^{n}(a_i+b_i)\gamma_i$$

$$= \sum_{i=1}^{n} a_i\gamma_i + \sum_{i=1}^{n} b_i\gamma_i$$

$$= \sigma(\alpha) + \sigma(\beta),$$

$$\sigma(k\alpha) = \sigma\left(\sum_{i=1}^{n}(ka_i)\alpha_i\right) = \sum_{i=1}^{n}(ka_i)\gamma_i$$

$$= k\sum_{i=1}^{n} a_i\gamma_i = k\sigma(\alpha),$$

σ V V' , $V \cong V'$. ×

1 , F n V F^n , V F^n : V $\alpha_1, \alpha_2, \dots, \alpha_n$; F^n $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$.

$$\sigma: V \to F^n$$

$$\alpha = \sum_{i=1}^{n} a_i\alpha_i \longmapsto \sum_{i=1}^{n} a_i\varepsilon_i = (a_1, a_2, \dots, a_n)', \qquad (4)$$

□□ V □□□□□□ α □□□□□ V □□□□ $\alpha_1, \alpha_2, \ldots, \alpha_n$ □□□□ $(a_1, a_2, \ldots, a_n)$□, □□□ V □ F^n □□□□□□□. □□□□□ F □□□ n □□□□□ V □ F^n □□, □□ V □ F^n □□□□□, □□□□□□□, □□□□□□□□□□□□□□□□. □□□□□□□□ F^n □□□□□□ F □ n □□□□□□□□.

□□ 2 □ V □□ F □ n □□□□□, U □ V □□□□□□. V □□□□□ $\alpha_1, \alpha_2, \ldots, \alpha_n$, σ □ V □□□□□□ α □□□□□□ $\alpha_1, \alpha_2, \ldots, \alpha_n$ □□□□. □

$$\sigma(U) \overset{\text{def}}{=} \{\sigma(\alpha) \mid \alpha \in U\},$$

□ $\sigma(U)$ □ F^n □□□□□□, □ $\dim U = \dim \sigma(U)$.

□□ □□ $\sigma(U)$ □□□□. □□ $\alpha, \beta \in U$, □ $\alpha + \beta \in U$, $k\alpha \in U$. □□ σ □ V □ F^n □□□□□□□, □□

$$\sigma(\alpha) + \sigma(\beta) = \sigma(\alpha + \beta) \in \sigma(U),$$

$$k\sigma(\alpha) = \sigma(k\alpha) \in \sigma(U),$$

□□ $\sigma(U)$ □ F^n □□□□□□.

□□ U □ $\sigma(U)$ □□□ F □□□□□□□□, □ σ □□□ U □□ U □ $\sigma(U)$ □□□□□□□, □□□□□ 1 □□□□□

$$\dim U = \dim \sigma(U).$$ x

□ 1 □ $\alpha_1, \alpha_2, \ldots, \alpha_n$ □□ F □□□□□ V □□□□, $\beta_1, \beta_2, \ldots, \beta_s$ □ V □□□□□□, □□

$$(\beta_1, \beta_2, \ldots, \beta_s) = (\alpha_1, \alpha_2, \ldots, \alpha_n) A, \tag{5}$$

□□ A □□□ $n \times s$ □□. □□:

$$\dim \langle \beta_1, \beta_2, \ldots, \beta_s \rangle = \operatorname{rank}(A). \tag{6}$$

□□ □ σ □□ V □ F^n □□□□□□□, □□ $\alpha \in V$ □□ α □□ $\alpha_1, \alpha_2, \ldots, \alpha_n$ □□□□, □ A □□□□□□: $A_1, A_2, \ldots, A_s$, □ (5) □□□, $\sigma(\beta_j) = A_j$, $j = 1, 2, \ldots, s$. □□□ 2 □,

$$\begin{aligned} \dim \langle \beta_1, \beta_2, \ldots, \beta_s \rangle &= \dim \sigma \langle \beta_1, \beta_2, \ldots, \beta_s \rangle \\ &= \dim \langle \sigma(\beta_1), \sigma(\beta_2), \ldots, \sigma(\beta_s) \rangle = \dim \langle A_1, A_2, \ldots, A_s \rangle \\ &= \operatorname{rank}(A). \end{aligned}$$ x

□□□□ F □□□□□□□□□□□□, □□□□□□(□□ V □□□□□□ V □ V □□□□□□□), □□□□□□□(□□□□□ 3 □□). □□□□□□□□□□□□□, □□□□□□□□.

□□ 1 □□, □□□ F □□□□□□□□□□□□□□□ S □□, □□□ 0 □□□□□($\Box$, $\{0\}$)□□□□□□□□□, □□ 1 □□□□□□□□□□□□

□□,□□2□□□□□□□□□□□□□,......□□□□□□□□□□□.□□□ F□□□□□□□□□□□□□□□□□□□□□□□□□□.□□□□,□□□□□□□□□□□□□□□!

□□3 □ F□□□□□□□□□□□□□□□□□□□□□□□□;□□□□□□□□□□□□□□□□.

□□ □σ□□ F□□□□□ V□ V′□□□□□□□.□□σ^{-1}□ V′□ V□□□□□.□□α',β'□ V′,□□□α,β□ V,□□$\alpha'=\sigma(\alpha)$,$\beta'=\sigma(\beta)$.□□$\sigma^{-1}(\alpha')=\alpha$,$\sigma^{-1}(\beta')=\beta$.□□

$$\begin{aligned}\sigma^{-1}(\alpha'+\beta') &= \sigma^{-1}(\sigma(\alpha)+\sigma(\beta)) = \sigma^{-1}[\sigma(\alpha+\beta)]\\ &= (\sigma^{-1}\sigma)(\alpha+\beta) = 1_V(\alpha+\beta)\\ &= \alpha+\beta = \sigma^{-1}(\alpha')+\sigma^{-1}(\beta');\end{aligned}$$

$$\sigma^{-1}(k\alpha') = \sigma^{-1}(k\sigma(\alpha)) = \sigma^{-1}(\sigma(k\alpha)) = \sigma^{-1}\sigma(k\alpha) = k\alpha = k\sigma^{-1}(\alpha').$$

□□σ^{-1}□ V′□ V□□□□□□□.

□σ□τ□□□□ F□□□□□ V□ V′□ V′□ V″□□□□□□□,□$\tau\sigma$□ V□ V″□□□□□,□□α,β□ V,k□ F,□

$$(\tau\sigma)(\alpha+\beta) = \tau(\sigma(\alpha+\beta)) = \tau[\sigma(\alpha)+\sigma(\beta)] = \tau\sigma(\alpha)+\tau\sigma(\beta),$$

$$(\tau\sigma)(k\alpha) = \tau(\sigma(k\alpha)) = \tau[k\sigma(\alpha)] = k\tau\sigma(\alpha).$$

□□$\tau\sigma$□ V□ V″□□□□□□□. X

□□ 8.3

1. □□:□□□ R□□□□□□□□□□□□□8.1□1□(2)□□□□□□ R^+□□.

2. □□:□ F□□□□□□□ $M_{s\times n}(F)$□ F^{sn}□□,□□□□□□□□□□□.

3. □□:□ F□□□ $< n$□□□□□□□□□□□□□□ $F[x]_n$□ F^n□□,□□□□□□□□□□.

4. □□ F□□□□□ V□□□□□$\alpha_1,\alpha_2,\alpha_3,\alpha_4$□□□□,□□:□□□$\alpha_1+\alpha_2,\alpha_2+\alpha_3,\alpha_3+\alpha_4,\alpha_4+\alpha_1$□□□□□□?□

$$W=\langle \alpha_1+\alpha_2,\alpha_2+\alpha_3,\alpha_3+\alpha_4,\alpha_4+\alpha_1\rangle,$$

□ W□□□□□□□.

5. □

$$L = \left\{ \begin{pmatrix} a & b\\ -b & a\end{pmatrix} \middle| a,b\in R \right\},$$

(1) □□ L□□□□□□ $M_2(R)$□□□□,□ L□□□□□□□;

(2) □□□□□ C□□□□□ R□□□□□□□ L□□,□□□□□□□□□□.

*6. □

$$H = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \middle| \alpha, \beta \in C \right\},$$

(1) □□ H□□□□□□，□□□□□□□□□□□□(□□□□□□□□□)□□□□□□□□□□□□□；

(2) □ H□□□□□□；

(3) □□ H□ R^4□□，□□□□□□□□□□.

□□□□□:□□□□□□□□

□ F□□□□□□，□□□□□ e. □□ F□□□□ 0，□ $e, 2e, \ldots, ne, \ldots$□ F□□□□□□□□，□□ F□□□□□□□，□□. □□ F□□□□□□□□ p. □

$$F_p = \{0, e, 2e, \ldots, (p-1)e\}.$$

□□ F_p□□ F□□□□□□□□，□□ F_p□ F□□□□□，□□ e□ F_p□□□□□□，□ F_p□□□□，□□ F_p□□□□□□ ie，□□ $(i, p) = 1$，□□□□ $u, v \in Z$，□□ $ui + vp = 1$. □□

$$e = 1e = uie + vpe = u(ie) = (ue)(ie).$$

□ $u = lp + r, \quad 0 \leq r < p$，□

$$ue = (lp + r)e = lpe + re = re \in F_p.$$

□□ ie□ F_p□□□□ re，□□ F_p□□□□.

□□□ F□□□□□ F_p□□□□□□，□□□□□ F□□□，□□□□□ F_p□□□□ F□□□□ F□□□. □□ F□□，□□□□□□□□□□□□□□□ 8□□□□□，□□ F□□ F_p□□□□□□，□□ F□□□□□□□□□，□□ F□□□□□□□，□□□□□ n，□ $F \cong F_p^n$. □□ F□ F_p^n□□□□□，□□ $|F| = |F_p^n|$. □□ $F_p^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in F_p, i = 1, 2, \ldots, n\}$，□□ $|F_p^n| = p \cdot p \cdots p = p^n$. □□ $|F| = p^n$，□□□□□□□

□□ 1 □□□□□□□□□□□□□ p□□□，□□ p□□□. x

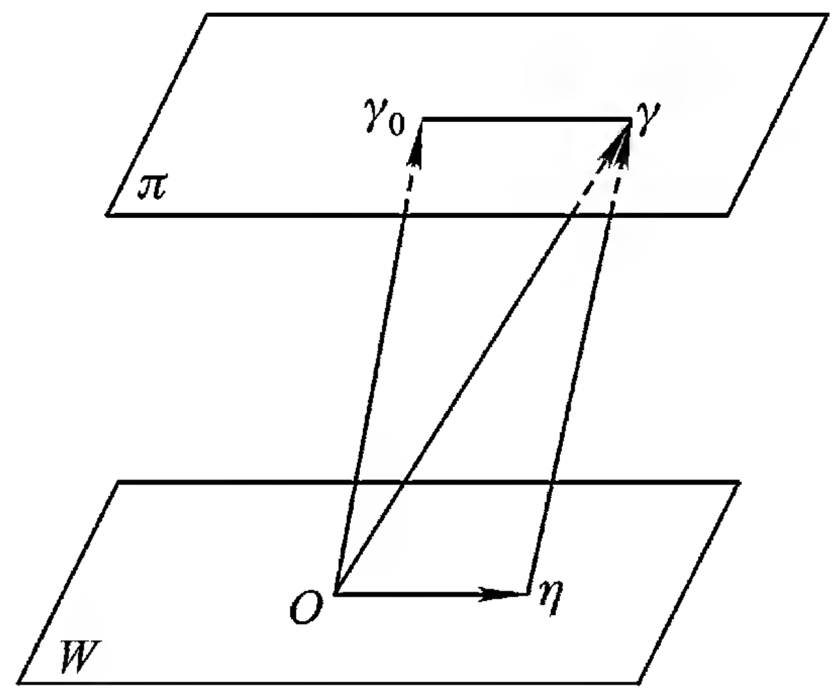
§4 □ □ □

□□□□□□□□□□□□ O□□□□□□□□□□□ 3□□□□□□ V，□□□□□□□□ W□ V□□□ 2□□□□，□ W□□□□□□□□□ π□□ V□□□□，□□ π□□□□□□□□□□□□□□□□. □□□□:π□

□□□□□□□?π□ W□□□□□?

□π□□□□□□□γ_0,□□ 8-3□□,π□□□□□□γ□□□□□□□□γ_0□ W□□□□□η□□:

$$\gamma=\gamma_0+\eta.$$



□ 8-3

□□,□□$\eta\in W$,□$\gamma_0+\eta\in\pi$,□□

$$\pi=\{\gamma_0+\eta\,|\,\eta\in W\}. \tag{1}$$

□(1)□□□□□□□□γ_0+W,□□□ W□□□□□,γ_0□□□□□□.□□

$$\gamma\in\gamma_0+W \Leftrightarrow \gamma=\gamma_0+\eta$$
$$\Leftrightarrow \gamma-\gamma_0=\eta\in W.$$

□□□□,□□□ V□□□□□□□□□□□□□:

$$\gamma\sim\gamma_0 \Leftrightarrow \gamma-\gamma_0\in W, \tag{2}$$

□□□□□□□□□□□□□□□□□□□□□□(□□ W□ V□□□□□□),□□□□□□□□,□□γ_0□□□□□□$\overline{\gamma_0}$□

$$\overline{\gamma_0}=\{\gamma\in V|\gamma\sim\gamma_0\}=\{\gamma\in V|\gamma-\gamma_0\in W\}$$
$$=\{\gamma\in V|\gamma=\gamma_0+\eta,\eta\in W\}$$
$$=\{\gamma_0+\eta\,|\,\eta\in W\}=\gamma_0+W.$$

□□□□□γ_0+W□□□□□$\overline{\gamma_0}$.

□□□□,□□□□□ V□,□ W□□□□□□□□□π□ W□□□□□,□□□□(2)□□□□□□□□□□□□□□□□□.

□□□,□ V□□ F□□□□□□□□, W□ V□□□□□□.□ V□□□□□□□□□□□□:

$$\alpha\sim\beta \Leftrightarrow \alpha-\beta\in W \tag{3}$$

□□□□□□□□□,□□□□:

□□$\alpha\in V$,□□$\alpha-\alpha=0\in W$,□□$\alpha\sim\alpha$.

□□$\alpha\sim\beta$,□$\alpha-\beta\in W$,□□

$$\beta-\alpha=-(\alpha-\beta)\in W,$$

□□$\beta\sim\alpha$.

□□□,□□□□□□□□□□□□(□□ W□□□□□).

α□□□□□$\overline{\alpha}$□

$$\overline{\alpha}=\{\beta\in V|\beta\sim\alpha\}=\{\beta\in V|\beta-\alpha\in W\}$$
$$=\{\beta\in V|\beta=\alpha+\gamma,\gamma\in W\}$$

$$=\{\alpha+\gamma \mid \gamma \in W\}, \qquad (4)$$

□(4)□□□□□□□□□ $\alpha+W$,□□□ W □□□□□,α □□□□□□.□
□ α □□□□□ α □□□ α □□□□ W □□□□□ $\alpha+W$.□□

$$\beta \in \alpha+W \Leftrightarrow \beta \sim \alpha \Leftrightarrow \beta-\alpha \in W. \qquad (5)$$

□□□□□□□□□5□§1□□□4□□

$$\alpha+W=\delta+W \Leftrightarrow \alpha-\delta \in W. \qquad (6)$$

□(6)□□□,□□□□ $\alpha+W$ □□□□□□□□;□□ $\alpha-\delta \in W$,□□ δ □□□□□□□□□□□□□.

□□□5□§1□□□4, V □□□□□(□ W □□□□□)□□□□□□□ V □□□□□□□□,□□ $V/$□.□□□□□□□□□□□□ W □□□,□□□ $V/$ □□□ V □□□□□ W □□□,□□ V/W,□

$$V/W=\{\alpha+W \mid \alpha \in V\}. \qquad (7)$$

□□□□ V □□□□□□□□□□□□□□□□,□□□□□□□□ V □□□□ W □□□ V/W □□□□□□□□□□□□□□□□.□□□□ V/W □□ V □□□,□□□□□ V □□□□□□□□□ V/W □□.

□□ 1 □ W □□ F □□□□□ V □□□□□□.□□□ V/W □□□

$$(\alpha+W)+(\beta+W) \overset{\text{def}}{=} (\alpha+\beta)+W, \qquad (8)$$

$$k(\alpha+W) \overset{\text{def}}{=} k\alpha+W, \quad k \in F, \qquad (9)$$

□(8)□□(9)□□□□□ V/W □□□□□□□□□□□□□□,□□ V/W □□□□□□□□□□ F □□□□□□,□□□□□ W(□ $0+W$),□□□□ V/W □□ V □ W □□□□.

□□ (8)□□(9)□□□□□□□□,□□□□□□□□□□□□□□□□□□□□□□.□

$$\alpha_1+W=\alpha+W, \qquad \beta_1+W=\beta+W.$$

□ $\alpha_1-\alpha \in W$, $\beta_1-\beta \in W$.□□

$$(\alpha_1+\beta_1)-(\alpha+\beta)=(\alpha_1-\alpha)+(\beta_1-\beta) \in W,$$

$$k\alpha_1-k\alpha=k(\alpha_1-\alpha) \in W,$$

□□ $$(\alpha_1+\beta_1)+W=(\alpha+\beta)+W, \quad k\alpha_1+W=k\alpha+W.$$

□□□□□□□□□□□□□□□□□□□□□□□□□.

□□□□□□□□□□□□□8□□□□□□ V/W □□□□.□□

$$(\alpha+W)+(0+W)=(\alpha+0)+W=\alpha+W, \quad \forall \alpha+W \in V/W,$$

□□ $0+W$(□ W)□ V/W □□□□.

□□□□,V/W □□□ F □□□□□□□. X

□□,□□□□□□□□□□ V □□□□,□ W □□□□□□□□□ W

□□□□□□□ V□□ W□□□□ V/W. □□,□ W□□□□□□□□□ W □ γ_0□ $-\gamma_0$□□□□□□(□□□ 8-3),□□□ W□□□□□□□□(□ V/W□□□□□□□)□□□ $k\gamma_0+W$□□□,□□ k□ R. □□

$$k\gamma_0+W=k(\gamma_0+W),$$

□□ V/W□□□□□□ $k\gamma_0+W$□□□□□ γ_0+W□□□□,□ γ_0+W□ □□□,□□ γ_0+W□□□□□ V/W□□□□,□□ $\dim(V/W)=1$. □□ $\dim V=3$, $\dim W=2$,□□□□, $\dim(V/W)=\dim V-\dim W$. □□□,□□□□ □□□:

□□ 2 □□ W□□ F□□□□□□□□ V□□□□□□,□

$$\dim(V/W)=\dim V-\dim W. \tag{10}$$

□□ □ W□□□□□ $\alpha_1,\dots,\alpha_s$,□□□□□ V□□□□

$$\alpha_1,\dots,\alpha_s,\alpha_{s+1},\dots,\alpha_n.$$

□□ $\beta+W$□ V/W,□ $\beta=b_1\alpha_1+\dots+b_n\alpha_n$,□

$$\begin{aligned}\beta+W&=(b_1\alpha_1+\dots+b_n\alpha_n)+W\\&=(b_1\alpha_1+W)+\dots+(b_s\alpha_s+W)\\&\quad+(b_{s+1}\alpha_{s+1}+W)+\dots+(b_n\alpha_n+W)\\&=W+\dots+W+b_{s+1}(\alpha_{s+1}+W)\\&\quad+\dots+b_n(\alpha_n+W)\\&=b_{s+1}(\alpha_{s+1}+W)+\dots+b_n(\alpha_n+W).\end{aligned}\tag{11}$$

□□□ V/W□□□□□□□□□ $\alpha_{s+1}+W,\dots,\alpha_n+W$□□□□□. □□□ $\alpha_{s+1}+W,\dots,\alpha_n+W$□□□□,□□□□ V/W□□□□,□□

$$\dim(V/W)=n-s=\dim V-\dim W.$$

□□□□

$$k_1(\alpha_{s+1}+W)+\dots+k_{n-s}(\alpha_n+W)=W, \tag{12}$$

□ $$(k_1\alpha_{s+1}+\dots+k_{n-s}\alpha_n)+W=W,$$

□□ $$k_1\alpha_{s+1}+\dots+k_{n-s}\alpha_n□W,$$

□□ $$k_1\alpha_{s+1}+\dots+k_{n-s}\alpha_n=l_1\alpha_1+\dots+l_s\alpha_s,$$

□ $$l_1\alpha_1+\dots+l_s\alpha_s-k_1\alpha_{s+1}-\dots-k_{n-s}\alpha_n=0. \tag{13}$$

□□ $\alpha_1,\dots,\alpha_s,\alpha_{s+1},\dots,\alpha_n$□□□□,□□

$$l_1=\dots=l_s=k_1=\dots=k_{n-s}=0$$

□□□□ $\alpha_{s+1}+W,\dots,\alpha_n+W$□ V/W□□□□□□□□□. x

□□□□□□□□□□□□□□□□ V□□□□□ W□□□□□,□□ □□ V/W□□□□. □□□□□,□□ 2□□□, $\dim(V/W)$□□□□□□□

□□□□□□.

□□ 1 □ W□□ F□□□□□ V□□□□□□,□□ V□ W□□□□ V/W□□□□□,□ dim(V/W)□□□□□ W□ V□□□□□(codimension),□□ $\mathrm{codim}_V W$□ codim W.

□ W□□ F□□□□□ V□□□□□□,□ V□□□□ V/W□□□□□□□□□$\pi: \alpha \longmapsto \alpha + W$,□□□□□□□□□□□□(canonical mapping).□□□□□□.

□□ 8.4

1. □ U, W□□□ F□□□□□ V□□□□,□□:□□ $V = U \oplus W$,□□

$$V/U \cong W.$$

2. □ V□□□□□,U□□□□ O□□□□□,□□□ V/U□□□□□□□?□ V/U□□□□□□.

3. □ V□□ F□□□□ n□□□□□,$n \geq 3$, U□ V□□□ 2□□□□,□ Ω_1□□ V□□□ U□□□ n-1□□□□□□□□□,□ Ω_2□□□□□ V/U□□□ n-3□□□□□□□□□.□

$$\begin{aligned} \sigma: \Omega_1 & \longrightarrow \Omega_2 \\ W & \longmapsto W/U, \end{aligned}$$

□□:σ□□□□.

4. □ U, W□□□ F□□□□□ V□□□□,□□

$$(U+W)/W \cong U/(U \cap W).$$

□□□□□:□□□

□□□□□□,□□□□□□□□□□.□□,□□□□□□□□□□□□□□□□□□□□□,□□□□□□□□□.□□□□□□□□□□□□□□□□□□□□□□□□,□□□□□□□□ 0,□□□□□□□□ 1.□□□□□□□□□□□□□□□□ 0□ 1□□□□□,□□□□□□□□□□.□□□□□□,□□□□□□□□□□□□□□□□□□□□□□□□□,□□□ 0□□ 1, 1□□ 0.□□□□□□□□□□□□□□□□□□□□□□.□□:□□□□□□□□□?□□□□□□□,□□□□□□?□□□□□□□□□□□□□□□□□□□□□□□□□.□□□□□□□□□□□□□□□□□□□□.

□□□□□□□□□□ 0□ 1□□□ 4□□□□,□□□□□ 4□□□□

□□□□□□ Z_2 □□ 4 □□□□□ Z_2^4 □□□□□□(a_1, a_2, a_3, a_4). □□□□
□□□□□□□□□□□□?□□□□□□□□□□□□□. □□, □□□□□
□, □□□□□□□□□□□□, □□□□□□□□□□□□, □□, communication
(□□)□□□□"conmunication", □□□□□□□□□□□"n"□□□. □□□
□□□□□□□□□□, □□□□□□□□□□□□□□. □□□□□□, □□□
□ 4 □□□(a_1, a_2, a_3, a_4)□□□□□ 3 □□□, □□ Z_2 □□ 7 □□□$(a_1, a_2,$
$a_3, a_4, c_1, c_2, c_3)$, □□

$$
\begin{aligned}
c_1 &= a_1 + a_2 + a_3,\\
c_2 &= a_1 + a_2 + a_4,\\
c_3 &= a_1 + a_3 + a_4.
\end{aligned}
\tag{1}
$$

□□□□□□□□□ Z_2 □□□□□□ Z_2^4 □ Z_2^7 □□□□□ σ:

$$
\begin{aligned}
\sigma:\quad & Z_2^4 \longrightarrow Z_2^7\\
& (a_1, a_2, a_3, a_4) \longmapsto (a_1, a_2, a_3, a_4, c_1, c_2, c_3),
\end{aligned}
\tag{2}
$$

□□ c_1, c_2, c_3 □(1)□□□, □□ σ □□□. □ σ □□□□□; σ □□ $\mathrm{Im}\,\sigma$ □□
□□□, □□□□ C; □ C □□□□□□□□□□□□□; □ σ □□□ Z_2^7 □□
□□□□□□□□□, □□$(a_1, a_2, a_3, a_4, c_1, c_2, c_3)$□□ 4 □□□□□□□
□, □ 3 □□□□□□□□. □□□□□□□□ C □□□□□□□□. □□□□
□□(1)□□□

$$
\begin{pmatrix} c_1\\ c_2\\ c_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0\\ 1 & 1 & 0 & 1\\ 1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_1\\ a_2\\ a_3\\ a_4 \end{pmatrix}.
\tag{3}
$$

□(3)□□□□ 3×4 □□□□ A, □(3)□□□□

$$
A \begin{pmatrix} a_1\\ a_2\\ a_3\\ a_4 \end{pmatrix} - I_3 \begin{pmatrix} c_1\\ c_2\\ c_3 \end{pmatrix} = 0.
\tag{4}
$$

(4)□□□□

$$
(A \quad -I_3) \begin{pmatrix} a_1\\ a_2\\ a_3\\ a_4\\ c_1\\ c_2\\ c_3 \end{pmatrix} = 0.
\tag{5}
$$

[illegible] (5) [illegible] $(A - I_3)$ [illegible] H, [illegible]

$$H = (A - I_3). \tag{6}$$

[illegible] (5) [illegible] (6) [illegible], [illegible] α [illegible] Z_2^7 ([illegible]), [illegible]

$$\alpha \in C \iff H\alpha = 0. \tag{7}$$

[illegible] α [illegible] α [illegible] $HX = 0$ [illegible]. [illegible] $HX = 0$ [illegible] W [illegible] Z_2^7 ([illegible]) [illegible], [illegible] C [illegible] Z_2^7 ([illegible]) [illegible]. [illegible] rank$(H) = 3$, [illegible] $\dim W = 7 - 3 = 4$. [illegible]

$$\dim C = 4. \tag{8}$$

[illegible] C [illegible] (7, 4) [illegible], [illegible] 7 [illegible] σ [illegible] Z_2^7 [illegible], 4 [illegible] C [illegible]. (6) [illegible] 3×7 [illegible] H [illegible] C [illegible].

[illegible] α, [illegible] γ. [illegible] $H\gamma$, [illegible] $H\gamma$ [illegible] 0, [illegible] γ [illegible]. [illegible], [illegible] γ [illegible] α? [illegible] ([illegible]) [illegible]: [illegible], [illegible], [illegible] C [illegible], [illegible] γ [illegible]. [illegible]:

[illegible] 1 [illegible] α, β [illegible] Z_2^n, α [illegible] β [illegible] α [illegible] β [illegible] Hamming [illegible], [illegible] $d(\alpha, \beta)$.

[illegible] $d(\alpha, \beta)$ [illegible] $\alpha - \beta$ [illegible], [illegible]:

[illegible] 2 [illegible] α [illegible] Z_2^n, α [illegible] α [illegible] **Hamming** [illegible], [illegible] $W(\alpha)$.

[illegible],

$$d(\alpha, \beta) = W(\alpha - \beta). \tag{9}$$

[illegible] γ [illegible], [illegible] C [illegible] γ [illegible] Hamming [illegible], [illegible] γ [illegible] Hamming [illegible] β, [illegible] γ [illegible] β. [illegible] σ [illegible], β [illegible] α. [illegible]. [illegible], [illegible], [illegible] γ [illegible] α [illegible]. [illegible]

$$e = \gamma - \alpha, \tag{10}$$

[illegible] e [illegible]. [illegible]

$$He = H(\gamma - \alpha) = H\gamma - H\alpha = H\gamma, \tag{11}$$

[illegible] $H\gamma$ [illegible] γ [illegible]. [illegible] (11) [illegible], [illegible] e [illegible] γ [illegible].

[illegible], α [illegible] β [illegible]

$$\iff H\alpha = H\beta$$

$$\Leftrightarrow \quad H(\alpha - \beta)^{T} = 0$$
$$\Leftrightarrow \quad \alpha - \beta \in C$$
$$\Leftrightarrow \quad \alpha + C = \beta + C. \qquad (12)$$

[illegible] α [illegible] β [illegible] α [illegible] β [illegible] C [illegible], [illegible], [illegible] e [illegible] $\gamma + C$. [illegible], [illegible] $\gamma + C$ [illegible] e. [illegible] γ [illegible] $\gamma - e$. [illegible].

[illegible], [illegible] C [illegible], [illegible]. [illegible], [illegible]; [illegible], [illegible], [illegible]. [illegible] γ [illegible], [illegible] $H\gamma^{T}$, [illegible], [illegible] γ, [illegible] γ [illegible], [illegible] γ [illegible].

[illegible], [illegible] C [illegible]. [illegible] C [illegible] H [illegible]

$$H = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

[illegible], $\mathrm{rank}(H) = 2$, [illegible] $HX = 0$ [illegible] W [illegible] $4 - 2 = 2$. [illegible] $\dim C = 2$. [illegible] C [illegible] $(4, 2)$ [illegible], [illegible] Z_2^4 / C [illegible]

$$\dim(Z_2^4 / C) = \dim Z_2^4 - \dim C = 4 - 2 = 2,$$

[illegible] Z_2^4 / C [illegible] $2^2 = 4$, [illegible] C [illegible] 4 [illegible]. [illegible] C [illegible] ([illegible] (a_1, a_2, a_3, a_4) [illegible] $a_1 a_2 a_3 a_4$):

[illegible]	[illegible]	[illegible]
0000	1010 0111 1101	$\begin{pmatrix}0\\0\end{pmatrix}$
1000	0010 1111 0101	$\begin{pmatrix}1\\0\end{pmatrix}$
0100	1110 0011 1001	$\begin{pmatrix}1\\1\end{pmatrix}$
0001	1011 0110 1100	$\begin{pmatrix}0\\1\end{pmatrix}$

[illegible], [illegible] $\gamma = 1110$, [illegible] $H\gamma^{T} = \begin{pmatrix}1\\1\end{pmatrix}$. [illegible] $\begin{pmatrix}1\\1\end{pmatrix}$, [illegible] 1110, [illegible] 1010, [illegible] γ [illegible] 1010.

[illegible] 9 [illegible]

[illegible] F [illegible]. [illegible], [illegible], [illegible]. [illegible]. [illegible].

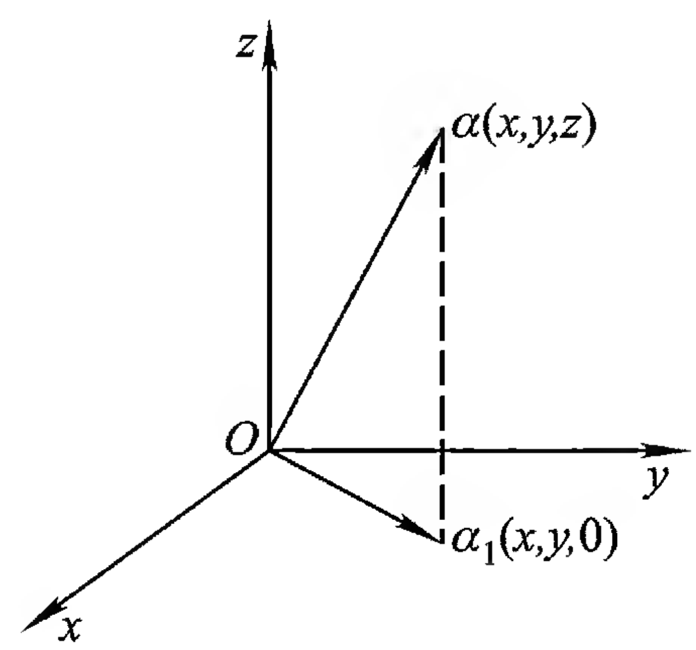
§ 1 [illegible]

[illegible] V [illegible], [illegible] $\alpha = (x, y, z)$ [illegible] $\alpha_1 = (x, y, 0)$ [illegible] V [illegible] V [illegible], [illegible] V [illegible] xOy [illegible] ([illegible] W) [illegible], [illegible] P_W, [illegible]

$$P_W(\alpha) = \alpha_1,$$

[illegible] 9-1 [illegible]. [illegible] P_W [illegible].

[illegible] $[a, b]$ [illegible] $C[a, b]$ [illegible], [illegible] R [illegible]. [illegible] R [illegible] $C[a, b]$ [illegible] R [illegible], [illegible]:



[illegible] 9-1

$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx,$$

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx,$$

[illegible], [illegible].

[illegible] 1 [illegible] V [illegible] V' [illegible] F [illegible], V [illegible] V' [illegible] A [illegible], [illegible]

$$A(\alpha + \beta) = A(\alpha) + A(\beta), \quad \forall\, \alpha, \beta \in V \tag{1}$$

$$A(k\alpha) = kA(\alpha), \quad \forall\, \alpha \in V, k \in F \tag{2}$$

[illegible] A [illegible] V [illegible] V' [illegible].

[illegible] V [illegible] V [illegible].

[illegible] F [illegible] V [illegible] F [illegible] V [illegible].

□□,□□□□ V □ xOy □□□□□□ P_W□ V□□□□□□□□.□□□□□□□ $C[a,b]$□□□□□□□□.

□ 1 □□□□ V □ V□□□□□(□ $A(\alpha)=0$, $\forall\alpha\in V$)□□□□□,□□ 0.

□ 2 □□□□ V□□□□□ 1_V□ V□□□□□□□□,□□□□ I.

□ 3 □□ k□ F, F□□□□□ V□□□□□□□□ $k(\alpha)=k\alpha$,□□ V□□ k□□□□□□□.□ $k=0$□,□□□□□□;□ $k=1$□,□□□□□□□.

□ 4 □ A□□ F□□□□ $s\times n$□□,□

$$A(\alpha)\overset{\text{def}}{=}A\alpha,\ \forall\alpha\in F^n,$$

□ A□ F^n□ F^s□□□□□□□.

□ 5 □ $C^{(1)}(a,b)$□□□□(a,b)□□□□□□□□□□□□□□,□□□□□□□□ R□□□□□□ $R^{(a,b)}$□□□□□□.□ D□□□□□,□

$$D(f(x))=f'(x),\ \forall f(x)\in C^{(1)}(a,b),$$

□ D□ $C^{(1)}(a,b)$□ $R^{(a,b)}$□□□□□□□.

□ 6 σ□□□□□ V□ V'□□□□□□□□□□□□□σ□ V□ V'□□□□□□□□.

□□ □□□□□□□□□□□□□□□□:□□,□□□□□□□□□□□□□□□□□□,□□□□□□.

X

□□□□□□□□□□□□□□□□□□□□,□□□□□□□□□□,□□□□□□□□□□□□□□□□□□,□□□□□□□□□□□□.□□□□ A□□□ K□□□□□ V□ V'□□□□□,□ A□□□□□:

1° $A(0)=0'$,□□□ $0'$□□ V'□□□□□;

2° $A(-\alpha)=-A(\alpha)$, $\forall\alpha\in V$;

3° $A(k_1\alpha_1+k_2\alpha_2+\dots+k_s\alpha_s)=k_1A(\alpha_1)+k_2A(\alpha_2)+\dots+k_sA(\alpha_s)$;

4° □□ $\alpha_1,\alpha_2,\dots,\alpha_s$□ V□□□□□□□□□□□,□ $A(\alpha_1)$, $A(\alpha_2)$, $\dots$, $A(\alpha_s)$□ V'□□□□□□□□□□□;□□□□□□□.□□□□□□□□□□□□□□□□□□□□□□□□□□□□;

5° □□ V□□□□□,□ $\alpha_1,\alpha_2,\dots,\alpha_n$□ V□□□□,□□□ V□□□□□ $\alpha=a_1\alpha_1+a_2\alpha_2+\dots+a_n\alpha_n$,□

$$A(\alpha)=a_1A(\alpha_1)+a_2A(\alpha_2)+\dots+a_nA(\alpha_n).\tag{3}$$

□□□,□□□□□ V□□□□$\alpha_1,\alpha_2,\dots,\alpha_n$□ A□□□,□□ V□□□□□□ A□□□□□□□□.□ n□□□□□ V□ V'□□□□□□□□□□ V□□□□□□□□□□.

□□□ F □□□□□□□□□ V □ V'，□□□□ V □ V' □□□□□□□？□□ V □□□□□，□□□□□□□□。□□□□□□□□：

□□ **1** □ V □ V' □□□ F □□□□□□，V □□□□ n，V □□□□□ $\alpha_1,\alpha_2,\ldots,\alpha_n$，$V'$ □□□□□ n □□□ $\gamma_1,\gamma_2,\ldots,\gamma_n$（□□□□□□□□□），□

$$A:\quad V \to V'$$

$$\alpha = \sum_{i=1}^{n} a_i\alpha_i \longmapsto \sum_{i=1}^{n} a_i\gamma_i, \tag{4}$$

□ A □ V □ V' □□□□□□□□，□ $A(\alpha_i)=\gamma_i$，$i=1,2,\ldots,n$.

□□ □□ $\alpha_1,\alpha_2,\ldots,\alpha_n$ □ V □□□□，□□ α □□□ $\alpha_1,\alpha_2,\ldots,\alpha_n$ □□□□□□□□□□，□□ A □ V □ V' □□□□□。□ V □□□□□□□

$$\alpha = \sum_{i=1}^{n} a_i\alpha_i,\qquad \beta = \sum_{i=1}^{n} b_i\alpha_i,$$

□

$$A(\alpha+\beta) = A\left(\sum_{i=1}^{n}(a_i+b_i)\alpha_i\right) = \sum_{i=1}^{n}(a_i+b_i)\gamma_i$$

$$= \sum_{i=1}^{n} a_i\gamma_i + \sum_{i=1}^{n} b_i\gamma_i = A(\alpha)+A(\beta),$$

$$A(k\alpha) = A\left(\sum_{i=1}^{n}(ka_i)\alpha_i\right) = \sum_{i=1}^{n}(ka_i)\gamma_i$$

$$= k\sum_{i=1}^{n} a_i\gamma_i = kA(\alpha),\qquad \forall k\in F.$$

□□ A □ V □ V' □□□□□□□□。□□□

$$A(\alpha_i) = A(0\alpha_1+\cdots+0\alpha_{i-1}+1\alpha_i+0\alpha_{i+1}+\cdots+0\alpha_n) = \gamma_i,$$

□□ $i=1,2,\ldots,n$. X

□□ V □ V' □□□□□□□□□□ V □□□□□□□□□□□，□□□□ 1 □□□ $A(\alpha_i)=\gamma_i$ $(i=1,2,\ldots,n)$ □□□□□□□□□。

□□ **2** □ V □□ F □□□□□□□□，U，W □ V □□□□□□，□

$$V = U \oplus W. \tag{5}$$

□□ $\alpha\in V$，□ $\alpha=\alpha_1+\alpha_2$，□□ $\alpha_1\in U$，$\alpha_2\in W$。□

$$P_U:\quad V \to V$$

$$\alpha=\alpha_1+\alpha_2 \longmapsto \alpha_1, \tag{6}$$

□ P_U □ V □□□□□□□□，□ P_U □□□□ W □ U □□□□，□□□

$$P_U(\alpha) = \begin{cases} \alpha, & □\ \alpha\in U, \\ 0, & □\ \alpha\in W; \end{cases} \tag{7}$$

□□(7)□□ V □□□□□□□□□□。

□□ □□ $V=U\oplus W$, □□ α □□□ U □□□□□ W □□□□□□ □□□□□□. □□ P_U □ V □ V □□□□□. □□ V □□□□□

$$\alpha=\alpha_1+\alpha_2,\qquad \beta=\beta_1+\beta_2,$$

□□ α_1,β_1 □ U, α_2,β_2 □ W. □ $\alpha_1+\beta_1$ □ U, $\alpha_2+\beta_2$ □ W. □□

$$P_U(\alpha+\beta)=P_U[(\alpha_1+\beta_1)+(\alpha_2+\beta_2)]=\alpha_1+\beta_1=P_U(\alpha)+P_U(\beta),$$

$$P_U(k\alpha)=P_U(k\alpha_1+k\beta_1)=k\alpha_1=kP_U(\alpha),\qquad \forall k\in F,$$

□□ P_U □ V □□□□□□□□.

□□ α □ U, □ $\alpha=\alpha+0$, □□ $P_U(\alpha)=\alpha$.

□□ α □ W, □ $\alpha=0+\alpha$, □□ $P_U(\alpha)=0$.

□ V □□□□□□ A □□□(7)□, □□ α □ V, □ $\alpha=\alpha_1+\alpha_2$, □□ α_1 □ U, α_2 □ W, □

$$A(\alpha)=A(\alpha_1+\alpha_2)=A(\alpha_1)+A(\alpha_2)=\alpha_1+0=\alpha_1=P_U(\alpha),$$

□□, $A=P_U$. ∎

□□□, □□ $P_W(\alpha)=\alpha_2$, □ P_W □□ V □□□□□□□□, □□□□□□ U □ W □□□□.

□□2□□□□, □□□□□□ V □□□□□□□□□□□□□: $V=U\oplus W$, □□ V □□□□ U(□ W)□□□□ P_U(□ P_W)□□ V □□□□□□□□□, □ P_U □□(7)□. □□□□□□□□□□□□□□□.

V □□□□ U(□ W)□□□□ P_U(□ P_W)□ V □ V □□□□□, □□□□□ □□, □ $P_U^2(=P_UP_U)$, P_UP_W, P_WP_U, P_W^2. □□ α □ V, □ $\alpha=\alpha_1+\alpha_2$, α_1 □ U, α_2 □ W. □

$$P_U^2(\alpha)=P_U(P_U(\alpha))=P_U(\alpha_1)=\alpha_1=P_U(\alpha),$$

$$P_UP_W(\alpha)=P_U(P_W(\alpha))=P_U(\alpha_2)=0,$$

$$P_WP_U(\alpha)=P_W(P_U(\alpha))=P_W(\alpha_1)=0,$$

□□

$$P_U^2=P_U,\qquad P_UP_W=P_WP_U=0 \tag{8}$$

□□□□ $P_W^2=P_W$.

V □□□□□□ A □□□□ $A^2=A$, □□ A □□□□□.

V □□□□□□□□ A, B □□□□ $AB=BA=0$, □□ A □ B □□□□.

□(8)□□□, □□ $V=U\oplus W$, □□□ P_U, P_W □□□□□□, □□ P_U □ P_W □□□□.

(8)□□□, □□□□□□□□□□□□□□□□□□□□□□□. □□□ □□□□□□□□□□□□□.

□□□□□ $\mathcal{A}\alpha$ □□ $\mathcal{A}(\alpha)$, □□□□□□□.

□□ 3 □ V, U, W □□□ F □□□□□□, $\mathcal{A}$ □ V □ U □□□□□□□, $\mathcal{B}$ □ U □ W □□□□□□□, □ $\mathcal{BA}$ □ V □ W □□□□□□□.

□□ □□ $\mathcal{BA}$ □ V □ W □□□□□. □□ α, β □ V, k □ F, □

$$
\begin{aligned}
(\mathcal{BA})(\alpha+\beta) &= \mathcal{B}(\mathcal{A}(\alpha+\beta)) = \mathcal{B}(\mathcal{A}\alpha+\mathcal{A}\beta) \\
&= \mathcal{B}(\mathcal{A}\alpha)+\mathcal{B}(\mathcal{A}\beta) = (\mathcal{BA})\alpha+(\mathcal{BA})\beta, \\
(\mathcal{BA})(k\alpha) &= \mathcal{B}(\mathcal{A}(k\alpha)) = \mathcal{B}(k\mathcal{A}\alpha) \\
&= k(\mathcal{B}(\mathcal{A}\alpha)) = k((\mathcal{BA})\alpha),
\end{aligned}
$$

□□ $\mathcal{BA}$ □ V □ V' □□□□□□□. x

□□□□□□□□□□□□, □□□□□, □□□□□□□□□□□□□□, □□□□□.

□□ 4 □ $\mathcal{A}$ □□□□□ V □ V' □□□□□□□, □□ $\mathcal{A}$ □□, □ $\mathcal{A}^{-1}$ □ V' □ V □□□□□□.

□ $\mathcal{A}$ □ V □ V' □□□□□□

⇒ $\mathcal{A}$ □ V □ V' □□□□□

⇒ $\mathcal{A}^{-1}$ □ V' □ V □□□□□

⇒ $\mathcal{A}^{-1}$ □ V' □ V □□□□□□□. x

□□□ F □□□□□□□□□ V □ V' □□□□□□□□□□□□□, □□ V □ V' □□□□□□□□□□□□□□□□ $\dim V = \dim V'$.

□□□□□□□□□□□□□□□□□.

□□ 5 □ $\mathcal{A}, \mathcal{B}$ □□□ F □□□□□ V □ V' □□□□□, k □ F, □

$$(\mathcal{A}+\mathcal{B})\alpha \overset{\text{def}}{=} \mathcal{A}\alpha+\mathcal{B}\alpha, \quad \forall \alpha \in V; \tag{9}$$

$$(k\mathcal{A})\alpha \overset{\text{def}}{=} k(\mathcal{A}\alpha), \quad \forall \alpha \in V, \tag{10}$$

□ $\mathcal{A}+\mathcal{B}, k\mathcal{A}$ □□ V □ V' □□□□□, □ $\mathcal{A}+\mathcal{B}$ □ $\mathcal{A}$ □ $\mathcal{B}$ □□, □ $k\mathcal{A}$ □ k □ $\mathcal{A}$ □□□□.

□□ □□(9)□□□ $\mathcal{A}+\mathcal{B}$ □ V □ V' □□□□□. □□□□ α, β □ V, l □ F, □

$$
\begin{aligned}
(\mathcal{A}+\mathcal{B})(\alpha+\beta) &= \mathcal{A}(\alpha+\beta)+\mathcal{B}(\alpha+\beta) \\
&= \mathcal{A}\alpha+\mathcal{A}\beta+\mathcal{B}\alpha+\mathcal{B}\beta \\
&= (\mathcal{A}+\mathcal{B})\alpha+(\mathcal{A}+\mathcal{B})\beta, \\
(\mathcal{A}+\mathcal{B})(l\alpha) &= \mathcal{A}(l\alpha)+\mathcal{B}(l\alpha) = l\mathcal{A}\alpha+l\mathcal{B}\alpha \\
&= l(\mathcal{A}\alpha+\mathcal{B}\alpha) = l(\mathcal{A}+\mathcal{B})\alpha,
\end{aligned}
$$

□□ $\mathcal{A}+\mathcal{B}$ □ V □ V' □□□□□.

□□□□□(10)□□□ $k\mathcal{A}$ □ V □ V' □□□□□. x

8 , F V V' F , $\mathrm{Hom}(V, V')$.

, F V F , $\mathrm{Hom}(V, V)$. $\mathrm{Hom}(V, V)$. , ; I : $IA = AI = A$, $\forall A \in \mathrm{Hom}(V, V)$. , :

$$A(B + C) = AB + AC,$$

$$(B + C)A = BA + CA.$$

$\mathrm{Hom}(V, V)$. , :

$$k(AB) = (kA)B = A(kB), \tag{11}$$

$k \in F$, $A, B \in \mathrm{Hom}(V, V)$.

2 A , F A , A F , A , A :

$$k(\alpha\beta) = (k\alpha)\beta = \alpha(k\beta), \quad \forall k \in F, \alpha, \beta \in A, \tag{12}$$

A F , A A .

, F V $\mathrm{Hom}(V, V)$, , F .

, F n $M_n(F)$, , F .

, A :

$$A^m \overset{\mathrm{def}}{=} \underbrace{A\,A\cdots A}_{m}. \tag{13}$$

A :

$$A^0 \overset{\mathrm{def}}{=} I. \tag{14}$$

:

$$A^m \cdot A^n = A^{m+n}, \quad (A^m)^n = A^{mn}, \quad m, n \in \mathbf{N}. \tag{15}$$

A ,

$$A^{-m} \overset{\mathrm{def}}{=} (A^{-1})^m, \quad m \in \mathbf{N}. \tag{16}$$

$f(x) = a_0 + a_1 x + \cdots + a_m x^m$ F , x V A ,

$$f(\mathcal{A}) = a_0 \mathcal{I} + a_1 \mathcal{A} + \dots + a_m \mathcal{A}^m. \tag{17}$$

□□, $f(\mathcal{A})$□□ V□□□□□□□□, □ $f(\mathcal{A})$□ $\mathcal{A}$□□□□□□. □□□□, □□□□ $\mathcal{A}$□□□□□□□□ $f(\mathcal{A})$□ $g(\mathcal{A})$□□□□□, □

$$f(\mathcal{A})\, g(\mathcal{A}) = g(\mathcal{A})\, f(\mathcal{A}). \tag{18}$$

□ V□□□□□ $\mathcal{A}$□□□□□□□□□□□□□ $F[\mathcal{A}]$. □□□□, $F[\mathcal{A}]$□□□□□□□□□□□□□□□, □□ $F[\mathcal{A}]$□□ $\mathrm{Hom}(V, V)$□□□□□, □ $F[\mathcal{A}]$□□□□, $\mathcal{I}$□ $F[\mathcal{A}]$. $F[\mathcal{A}]$□□□□□□□□□□□□□ $F[\mathcal{A}]$□□□□□, □□□ F□□□□□□□□□□□□ $\tau: k \longmapsto k$, τ □□□□□□□□□, □□ $F[\mathcal{A}]$□□□□ F□□□□□.

□□□□□□□□□, □□□□□□□□□□□□□□: □□, □ $V = U \oplus W$, □□ $\alpha \in V$, □

$$\alpha = \alpha_1 + \alpha_2, \quad \alpha_1 \in U, \alpha_2 \in V,$$

□
$$(\mathcal{P}_U + \mathcal{P}_W)\alpha = \mathcal{P}_U\alpha + \mathcal{P}_W\alpha = \alpha_1 + \alpha_2 = \alpha = \mathcal{I}\alpha,$$

□□
$$\mathcal{P}_U + \mathcal{P}_W = \mathcal{I}. \tag{19}$$

(18)□(19)□□□, □□ $V = U \oplus W$, □□□ $\mathcal{P}_U$□ $\mathcal{P}_W$□□□□□□□□□, □ $\mathcal{P}_U + \mathcal{P}_W$□□□□□□ $\mathcal{I}$. □□□□□□□□.

□ □ 9 1

1 □□□□□□□ K^3□□□□, □□□□□□□ ?

(1) $\mathcal{A}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ x_2 + x_3 \\ x_3^2 \end{pmatrix}$; (2) $\mathcal{A}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 - x_2 \\ x_2 + x_3 \\ 3x_1 - x_2 + x_3 \end{pmatrix}$.

2 □□□□□□□ $M_n(K)$□□□□, □□□□□□□ ?

(1) □ $A \in M_n(K)$, □

$$\mathcal{A}(X) = XA, \quad \forall X \in M_n(K);$$

(2) □ $B, C \in M_n(K)$, □

$$\mathcal{A}(X) = BXC, \quad \forall X \in M_n(K).$$

3 □□□□□□ $K[x]$□□□□□□□□□□ ? □□ $a \in K$, □

$$\mathcal{A}f(x) = f(x + a), \ \forall f(x) \in K[x].$$

4 □ $\mathbf{R}^+$□□□ 8 1□ 1□ (2)□□□□□□□, □□ $\mathbf{R}^+$□ $\mathbf{R}$□□□□□□□□□□□ ? □ $a > 0$□ $a \neq 1$, □

$$\log_a: \mathbf{R}^+ \to \mathbf{R}$$
$$x \longmapsto \log_a x.$$

*5. □ V□ $K[x, y]$□□□ m□□□□□□□□□□,□□□□□□□□□,□□□□□□□□□□□□□□ K□□□□□□□□.□□□□ K□□□□ 2□□□ $A = (a_{ij})$,□□ V□□□□□□□□ $\mathbf{A}$□□:

$$(\mathbf{A}f)(x, y) \overset{\text{def}}{=} f(a_{11}x + a_{21}y, a_{12}x + a_{22}y),$$

□□ $\mathbf{A}$□□□ V□□□□□□□□?

*6. □ 2^m□□□□□□□ F_{2^m}□□ F_2□□□□□□,□

$$f(x) = x^2,$$

□□ f□□□ F_{2^m}□□□□□□□□?

7. □ $K[x]$□,□

$$\mathbf{A}f(x) = xf(x), \quad \forall f(x) \in K[x].$$

(1) □□ $\mathbf{A}$□ $K[x]$□□□□□□□□;

(2) □ $\mathbf{D}$□□□□□,□□:

$$\mathbf{DA} - \mathbf{AD} = \mathbf{I}.$$

8. □ $\alpha_1, \alpha_2, \ldots, \alpha_n$□□□□□ V□□□□,$\mathbf{A}$□ V□□□□□□□□,□□:$\mathbf{A}$□□□□□□ $\mathbf{A}\alpha_1, \mathbf{A}\alpha_2, \ldots, \mathbf{A}\alpha_n$□ V□□□□.

9. □ $\mathbf{A}$□□□□□ V□□□□□□□□,□□:□□

$$\mathbf{A}^{m-1}\alpha \neq 0, \quad \mathbf{A}^m\alpha = 0 \quad (m > 0),$$

□ $\alpha, \mathbf{A}\alpha, \mathbf{A}^2\alpha, \ldots, \mathbf{A}^{m-1}\alpha$□□□□.

10. □ $\mathbf{A}, \mathbf{B}$□ V□□□□□,□□ $\mathbf{AB} - \mathbf{BA} = \mathbf{I}$,□□:

$$\mathbf{A}^k\mathbf{B} - \mathbf{B}\mathbf{A}^k = k\mathbf{A}^{k-1}, \quad k \geq 1.$$

11. □ V□□ F□□□□□□□,char $F \neq 2$.□ $\mathbf{A}, \mathbf{B}$□ V□□□□□,□□:

(1) $\mathbf{A} + \mathbf{B}$□□□□□□□□□ $\mathbf{AB} = \mathbf{BA} = 0$;

(2) □□ $\mathbf{AB} = \mathbf{BA}$,□ $\mathbf{A} + \mathbf{B} - \mathbf{AB}$□□□□□.

§2 □□□□□□□□

□□□□ V□ xOy□□□□□□ $\mathbf{P}_W$□ z□□□□□□□□ $\alpha = (0, 0, z)$□□□□□ $0 = (0, 0, 0)$,$\mathbf{P}_W$□□□ xOy□□,□□□ 9-1.□□□□□□□□□:

□□1 □ V□ V'□□□ F□□□□□,$\mathbf{A}$□ V□ V'□□□□□□,□□□ V'□□□□ 0□ $\mathbf{A}$□□□□□□□ $\mathbf{A}$□□,□□ $\mathrm{Ker}\,\mathbf{A}$,□

$$\mathrm{Ker}\,\mathbf{A} \overset{\text{def}}{=} \{\alpha \in V \mid \mathbf{A}\alpha = 0\}; \tag{1}$$

□□□□□ $\mathbf{A}$□□(□□□ $\mathbf{A}$□□□)□□ $\mathrm{Im}\mathbf{A}$□ $\mathbf{A}V$.

□□1 □ $\mathbf{A}$□□ F□□□□□ V□ V'□□□□□□,□ $\mathrm{Ker}\,\mathbf{A}$□ V□□□□□,$\mathrm{Im}\mathbf{A}$□ V'□□□□□.

[illegible] [illegible] $\mathcal{A}(0)=0$, [illegible] 0 [illegible] $\operatorname{Ker}\mathcal{A}$. [illegible] α,β [illegible] $\operatorname{Ker}\mathcal{A}$, k [illegible] F, [illegible]

$$\mathcal{A}(\alpha+\beta)=\mathcal{A}\alpha+\mathcal{A}\beta=0+0=0,$$
$$\mathcal{A}(k\alpha)=k\mathcal{A}\alpha=k0=0,$$

[illegible] $\operatorname{Ker}\mathcal{A}$ [illegible] V [illegible].

[illegible], $\operatorname{Im}\mathcal{A}$ [illegible] V [illegible]. [illegible] $\operatorname{Im}\mathcal{A}$ [illegible] γ_1,γ_2, [illegible] α_1,α_2 [illegible] V, [illegible] $\gamma_1=\mathcal{A}\alpha_1$, $\gamma_2=\mathcal{A}\alpha_2$, [illegible]

$$\gamma_1+\gamma_2=\mathcal{A}\alpha_1+\mathcal{A}\alpha_2=\mathcal{A}(\alpha_1+\alpha_2)\ [illegible]\ \operatorname{Im}\mathcal{A},$$
$$k\gamma_1=k\mathcal{A}\alpha_1=\mathcal{A}(k\alpha_1)\ [illegible]\ \operatorname{Im}\mathcal{A},\quad \forall k\ [illegible]\ F,$$

[illegible] $\operatorname{Im}\mathcal{A}$ [illegible] V [illegible]. X

[illegible] **2** [illegible] $\mathcal{A}$ [illegible] F [illegible] V [illegible] V [illegible], [illegible]

(i) $\mathcal{A}$ [illegible] $\operatorname{Ker}\mathcal{A}=0$;

(ii) $\mathcal{A}$ [illegible] $\operatorname{Im}\mathcal{A}=V$ [illegible].

[illegible] (i) [illegible]. [illegible] $\mathcal{A}$ [illegible]. [illegible] α [illegible] $\operatorname{Ker}\mathcal{A}$, [illegible]

$$\mathcal{A}(\alpha)=0=\mathcal{A}(0).$$

[illegible] $\alpha=0$, [illegible] $\operatorname{Ker}\mathcal{A}=0$.

[illegible]. [illegible] $\operatorname{Ker}\mathcal{A}=0$. [illegible] α_1,α_2 [illegible] V, [illegible] $\mathcal{A}\alpha_1=\mathcal{A}\alpha_2$. [illegible]

$$0=\mathcal{A}\alpha_2-\mathcal{A}\alpha_1=\mathcal{A}(\alpha_2-\alpha_1),$$

[illegible] $\alpha_2-\alpha_1$ [illegible] $\operatorname{Ker}\mathcal{A}$. [illegible] $\operatorname{Ker}\mathcal{A}=0$, [illegible] $\alpha_2-\alpha_1=0$. [illegible] $\alpha_1=\alpha_2$. [illegible] $\mathcal{A}$ [illegible].

(ii) [illegible]. X

[illegible] V [illegible] xOy [illegible] $\mathcal{P}_W$ [illegible] z [illegible], $\mathcal{P}_W$ [illegible] xOy [illegible]. [illegible]

$$\dim\operatorname{Ker}\mathcal{P}_W+\dim\operatorname{Im}\mathcal{P}_W=1+2=3=\dim V.$$

[illegible], [illegible]:

[illegible] **3** [illegible] V [illegible] V [illegible] F [illegible], [illegible] V [illegible], [illegible] $\mathcal{A}$ [illegible] V [illegible] V [illegible], [illegible] $\operatorname{Ker}\mathcal{A}$, $\operatorname{Im}\mathcal{A}$ [illegible], [illegible]

$$\dim\operatorname{Ker}\mathcal{A}+\dim\operatorname{Im}\mathcal{A}=\dim V. \tag{2}$$

[illegible] [illegible] V [illegible], [illegible] $\operatorname{Ker}\mathcal{A}$ [illegible]. [illegible] $\operatorname{Ker}\mathcal{A}$ [illegible] $\alpha_1,\dots,\alpha_m$, [illegible] V [illegible]

$$\alpha_1,\dots,\alpha_m,\alpha_{m+1},\dots,\alpha_n.$$

[illegible] $\operatorname{Im}\mathcal{A}$ [illegible] $\mathcal{A}\alpha$, [illegible] α [illegible] V. [illegible] $\alpha=\sum_{i=1}^{n}a_i\alpha_i$, [illegible]

$$\mathcal{A}\alpha=\sum_{i=1}^{n}a_i\mathcal{A}\alpha_i=a_{m+1}\mathcal{A}\alpha_{m+1}+\dots+a_n\mathcal{A}\alpha_n, \tag{3}$$

[illegible]

$$\operatorname{Im}\mathcal{A}=\langle\mathcal{A}\alpha_{m+1},\dots,\mathcal{A}\alpha_n\rangle, \tag{4}$$

□□ ImA □□□□□. □□□□ $A\alpha_{m+1},\dots,A\alpha_n$ □□□□:

□□ $$k_{m+1}A\alpha_{m+1}+\dots+k_nA\alpha_n=0,$$

□ $$A(k_{m+1}\alpha_{m+1}+\dots+k_n\alpha_n)=0,$$

□□ $$k_{m+1}\alpha_{m+1}+\dots+k_n\alpha_n\in \mathrm{Ker}A,$$

□□ $$k_{m+1}\alpha_{m+1}+\dots+k_n\alpha_n=l_1\alpha_1+\dots+l_m\alpha_m,$$

□ $$-l_1\alpha_1-\dots-l_m\alpha_m+k_{m+1}\alpha_{m+1}+\dots+k_n\alpha_n=0,$$

□□□□ $$l_1=\dots=l_m=k_{m+1}=\dots=k_n=0.$$

□□ $A\alpha_{m+1},\dots,A\alpha_n$ □□□□, □□□□ ImA □□□□. □□,

$$\dim \mathrm{Im}A=n-m=\dim V-\dim \mathrm{Ker}A.$$ x

□(3)□□□, □ V □□□□□, □ $\alpha_1,\dots,\alpha_n$ □ V □□□□,

□ $$\mathrm{Im}A=\langle A\alpha_1,\dots,A\alpha_n\rangle. \qquad (5)$$

□ V □□□□□, V □ V□□□□□□ A □□□□□□□□ A □□□, A □□ ImA □□□□□ A □□.

□□ 4 □□ V □ V□□□□ F □ n □□□□□, □ A □ V □ V□□□□□□□□, □ A □□□□□□□ A □□□.

□□ $$A\text{ □□□} \Leftrightarrow \mathrm{Ker}A=0$$
$$\Leftrightarrow \dim \mathrm{Im}A=\dim V=\dim V□$$
$$\Leftrightarrow \mathrm{Im}A=V□ \Leftrightarrow A\text{ □□□}.$$ x

□□ 5 □□□□□□□□□ V □□□□□ A, □□□□□□□□□□□ □□. x

□□, □□□□□□□□□ V □□□□□ A, □□□□□ $\mathrm{Ker}A$ □ $\mathrm{Im}A$ □□□□□□□ $\dim V$, □□ $\mathrm{Ker}A+\mathrm{Im}A$ □□□□□□□□□ V. □□, □□□□□ $K[x]_n$ □, □□□ D □□ $\mathrm{Im}D=K[x]_{n-1}$, D □□ $\mathrm{Ker}D=K$. □□

$$K+K[x]_{n-1}\text{□}K[x]_n.$$

□ A □□ F □□□ $s\times n$ □□, □

$$A: F^n \to F^s$$
$$\alpha\longmapsto A\alpha,$$

□ A □ F^n □ F^s □□□□□□□(□§1□□4). □□□□□□□□ 4□§7□, □□□□(□□ F □□□□□□): $\mathrm{Ker}A$ □□□□□□□□□ $AX=0$ □□□□; $\mathrm{Im}A$ □□□□ A □□□□. □□□□□□□□□ F □□□.

□ □ 9 2

1 □ V □□ F □□□□□□□□. U, W □ V □□□□□□. □ $V=U\oplus W$, □ P_U □

□□□□ W□ U□□□□.□□: $\operatorname{Ker} P_U = W, \operatorname{Im} P_U = U$.

2 □ V□□ F□□□□□□□□□, $\mathcal{A}$□ V□□□□□□□□□.□□:□□ $\mathcal{A}$□ V□□□□□□,□

$$V = \operatorname{Ker} \mathcal{A} \oplus \operatorname{Im} \mathcal{A},$$

□□ $\mathcal{A}$□□□□ $\operatorname{Ker} \mathcal{A}$□ $\operatorname{Im} \mathcal{A}$□□□□.

3 □ V, U, W□□□ F□□□□□□,□□ V□□□□□.□ $\mathcal{A}$□ V□ U□□□□□□□, $\mathcal{B}$□ U□ W□□□□□□□.□□:

$$\dim \operatorname{Ker}(\mathcal{B}\mathcal{A}) \leq \dim \operatorname{Ker} \mathcal{A} + \dim \operatorname{Ker} \mathcal{B}.$$

4 □ V, U, W□□□ F□□□□□□,□□ $\dim V = n, \dim U = m$,□ $\mathcal{A}$□ V□ U□□□□□□□, $\mathcal{B}$□ U□ W□□□□□□□.□□:

$$\operatorname{rank}(\mathcal{B}\mathcal{A}) \geq \operatorname{rank} \mathcal{A} + \operatorname{rank} \mathcal{B} - m$$

5 □ $\mathcal{A}, \mathcal{B}$□□□ F□□□□□ V□□□□□□,□□:

(1) $\mathcal{A}$□ $\mathcal{B}$□□□□□□□□□ $\mathcal{A}\mathcal{B} = \mathcal{B}, \mathcal{B}\mathcal{A} = \mathcal{A}$;

(2) $\mathcal{A}$□ $\mathcal{B}$□□□□□□□□□ $\mathcal{A}\mathcal{B} = \mathcal{A}, \mathcal{B}\mathcal{A} = \mathcal{B}$.

6 □ V□ V′□□□ F□□□□□□□□□□, $\mathcal{A}$□ V□ V′□□□□□□□.□□:□□□□□□ $V = U \oplus W, V' = M \oplus N$,□□ $\operatorname{Ker} \mathcal{A} = U$,□ $W \cong M$.

*7. □ V□□ F□□□□□□□□□, $\operatorname{char} F = 0$.□□:□□ $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_s$□ V□ s□□□□□□□□□,□□ V□□□□□□□□α,□□ $\mathcal{A}\alpha_1, \mathcal{A}\alpha_2, \ldots, \mathcal{A}\alpha_s$□□□□.

§3 □□□□□□□□□

□□□□ V□ xOy□□□□□□□ P_W□ V□□□□□□□□.□ x□□y□□z□□□□□□□□□□ e_1, e_2, e_3.□

$$P_W(e_1) = e_1,\ P_W(e_2) = e_2,\ P_W(e_3) = 0.$$

□□□□

$$(P_W(e_1), P_W(e_2), P_W(e_3)) = (e_1, e_2, e_3)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

□□□□□ 3□□□□□□□□□ P_W.□□□□□□,□□□□□□□□□□□□□□.

□ V□□ F□□ n□□□□□, $\mathcal{A}$□ V□□□□□□□□.□□□□, $\mathcal{A}$□□□ V□□□□□□□□□□.□□□ V□□□□ $\alpha_1, \alpha_2, \ldots, \alpha_n$.□□ $\mathcal{A}\alpha_i$□ V,□□ $\mathcal{A}\alpha_i$□□□ V□□□□ $\alpha_1, \alpha_2, \ldots, \alpha_n$□□□□□□□:

$$\begin{aligned}\mathcal{A}\alpha_1 &= a_{11}\alpha_1 + a_{21}\alpha_2 + \dots + a_{n1}\alpha_n,\\ \mathcal{A}\alpha_2 &= a_{12}\alpha_1 + a_{22}\alpha_2 + \dots + a_{n2}\alpha_n,\\ &\cdots\cdots\cdots\cdots\\ \mathcal{A}\alpha_n &= a_{1n}\alpha_1 + a_{2n}\alpha_2 + \dots + a_{nn}\alpha_n.\end{aligned} \tag{1}$$

□□□□□□, (1)□□□□□

$$(\mathcal{A}\alpha_1, \mathcal{A}\alpha_2, \dots, \mathcal{A}\alpha_n) = (\alpha_1, \alpha_2, \dots, \alpha_n)\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n}\\ a_{21} & a_{22} & \dots & a_{2n}\\ \dots & \dots & & \dots\\ a_{n1} & a_{n2} & \dots & a_{nn}\end{pmatrix}. \tag{2}$$

□□□(2)□□□□ n □□□(a_{ij})□□ A, □ A □□□□□□ $\mathcal{A}$ □□ $\alpha_1, \alpha_2, \dots, \alpha_n$ □□□□. A □□ j □□ $\mathcal{A}\alpha_j$ □□ $\alpha_1, \alpha_2, \dots, \alpha_n$ □□□□, $j = 1, 2, \dots, n$. □□ A □□□□□ $\mathcal{A}$ □□□□.

□□□$(\mathcal{A}\alpha_1, \mathcal{A}\alpha_2, \dots, \mathcal{A}\alpha_n)$□□□ $\mathcal{A}(\alpha_1, \alpha_2, \dots, \alpha_n)$. □□□(2)□□, n □□□ $\mathcal{A}$ □ V □□□□□ A □□ $\alpha_1, \alpha_2, \dots, \alpha_n$ □□□□□□□□□□□□:

$$\mathcal{A}(\alpha_1, \alpha_2, \dots, \alpha_n) = (\alpha_1, \alpha_2, \dots, \alpha_n) A. \tag{3}$$

□ **1** □ R^R □, □

$$V = □1, \sin x, \cos x□.$$

□□□ $\mathcal{D}$ □ V □□□□□□(□□ $\mathcal{D}(k_1 \cdot 1 + k_2 \sin x + k_3 \cos x) = k_2 \cos x - k_3 \sin x$ □ V); $1, \sin x, \cos x$ □ V □□□□(□□□□□□□□). □□ $\mathcal{D}$ □□ $1, \sin x, \sin x$ □□□□.

□ □□

$$\begin{aligned}\mathcal{D}(1) &= 0 = 0\cdot 1 + 0\cdot\sin x + 0\cdot\cos x,\\ \mathcal{D}(\sin x) &= \cos x = 0\cdot 1 + 0\cdot\sin x + 1\cdot\cos x,\\ \mathcal{D}(\cos x) &= -\sin x = 0\cdot 1 + (-1)\sin x + 0\cdot\cos x,\end{aligned}$$

□□ $\mathcal{D}$ □□ $1, \sin x, \cos x$ □□□□□

$$D = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}.$$

□□□□, n □□□□□ V □□□□□□□□□□□□□□. □□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□.

□ V □ V'□□□□ F □ n □□ s □□□□□, $\mathcal{A}$ □ V □ V'□□□□□□□. □ V □□□□□ $\alpha_1, \alpha_2, \dots, \alpha_n$; □ V'□□□□□ $\eta_1, \eta_2, \dots, \eta_s$. □□ $\mathcal{A}\alpha_i$ □ V', □□ $\mathcal{A}\alpha_i$ □□□ V'□□ $\eta_1, \eta_2, \dots, \eta_s$ □□□□□□□:

$$\begin{aligned}
\mathcal{A}\alpha_1 &= a_{11}\eta_1 + a_{21}\eta_2 + \cdots + a_{s1}\eta_s,\\
\mathcal{A}\alpha_2 &= a_{12}\eta_1 + a_{22}\eta_2 + \cdots + a_{s2}\eta_s,\\
&\cdots\cdots\cdots\cdots\\
\mathcal{A}\alpha_n &= a_{1n}\eta_1 + a_{2n}\eta_2 + \cdots + a_{sn}\eta_s.
\end{aligned} \tag{4}$$

, (4)

$$(\mathcal{A}\alpha_1, \mathcal{A}\alpha_2, \cdots, \mathcal{A}\alpha_n) = (\eta_1, \eta_2, \cdots, \eta_s)\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \cdots & \cdots & & \cdots\\ a_{s1} & a_{s2} & \cdots & a_{sn}\end{pmatrix}. \tag{5}$$

(5) $s\times n$ (a_{ij}) A, A $\mathcal{A}$ V $\alpha_1, \alpha_2, \cdots, \alpha_n$ V' $\eta_1, \eta_2, \cdots, \eta_s$. A j $\mathcal{A}\alpha_j$ $\eta_1, \eta_2, \cdots, \eta_s$, $j = 1, 2, \cdots, n$, A $\mathcal{A}$. (5) $s\times n$ A V V' $\mathcal{A}$ V $\alpha_1, \alpha_2, \cdots, \alpha_n$ V' $\eta_1, \eta_2, \cdots, \eta_s$:

$$\mathcal{A}(\alpha_1, \alpha_2, \cdots, \alpha_n) = (\eta_1, \eta_2, \cdots, \eta_s)A. \tag{6}$$

, F n V s V' $\mathcal{A}$ $s\times n$ A . 1 , V V' $\mathrm{Hom}(V, V')$ F . , F $s\times n$ $M_{s\times n}(F)$ F . : ?

V $\alpha_1, \alpha_2, \cdots, \alpha_n$; V' $\eta_1, \eta_2, \cdots, \eta_s$,

$$\begin{aligned}\sigma: \mathrm{Hom}(V, V') &\to M_{s\times n}(F)\\ \mathcal{A} &\longmapsto A,\end{aligned}$$

A $\mathcal{A}$ V $\alpha_1, \alpha_2, \cdots, \alpha_n$ V' $\eta_1, \eta_2, \cdots, \eta_s$.

, σ $\mathrm{Hom}(V, V')$ $M_{s\times n}(F)$. σ ? V V' $\mathcal{B}$ V $\alpha_1, \cdots, \alpha_n$ V' $\eta_1, \cdots, \eta_s$ B, $\sigma(\mathcal{B}) = B$. $\sigma(\mathcal{A}) = A$. $A = B$, A j $\mathcal{A}\alpha_j$ V' $\eta_1, \eta_2, \cdots, \eta_s$, B j $\mathcal{B}\alpha_j$, $\mathcal{A}\alpha_j = \mathcal{B}\alpha_j$, $j = 1, 2, \cdots, n$. α V, $\mathcal{A}\alpha = \mathcal{B}\alpha$. $\mathcal{A} = \mathcal{B}$, σ .

σ . C $M_{s\times n}(F)$,

$$(\gamma_1, \gamma_2, \cdots, \gamma_n) = (\eta_1, \eta_2, \cdots, \eta_s)C, \tag{7}$$

$\gamma_1, \gamma_2, \cdots, \gamma_n$ V'. §1 1, V V' C,

□□ $C\alpha_j=\gamma_j$, $j=1,2,\ldots,n$, □□

$$C(\alpha_1,\alpha_2,\ldots,\alpha_n)=(\gamma_1,\gamma_2,\ldots,\gamma_n)=(\eta_1,\eta_2,\ldots,\eta_s)C. \tag{8}$$

(8)□□□, C□□□□□ C□□□, □□ $\sigma(C)=C$. □□□ σ□□□.

□□□□σ□ $\mathrm{Hom}(V,V')$□ $M_{s\times n}(F)$□□□□□. □□□□σ□□□□□□□□□□□□□.

□ A,B□$\mathrm{Hom}(V,V')$, k□F. □ $\sigma(A)=A$, $\sigma(B)=B$. □□

$$\begin{aligned}
&(A+B)(\alpha_1,\alpha_2,\ldots,\alpha_n)\\
&=(A\alpha_1+B\alpha_1,\ A\alpha_2+B\alpha_2,\ldots,\ A\alpha_n+B\alpha_n)\\
&=(A\alpha_1,\ A\alpha_2,\ldots,\ A\alpha_n)+(B\alpha_1,\ B\alpha_2,\ldots,\ B\alpha_n)\\
&=(\eta_1,\eta_2,\ldots,\eta_s)A+(\eta_1,\eta_2,\ldots,\eta_s)B\\
&=(\eta_1,\eta_2,\ldots,\eta_s)(A+B),
\end{aligned}$$

□□□□□ $A+B$□□□□ $A+B$. □□

$$\sigma(A+B)=A+B=\sigma(A)+\sigma(B),$$

□□□σ□□□□□□. □□□□

$$\begin{aligned}
&(kA)(\alpha_1,\alpha_2,\ldots,\alpha_n)\\
&=(kA\alpha_1,\ kA\alpha_2,\ldots,\ kA\alpha_n)\\
&=k(A\alpha_1,\ A\alpha_2,\ldots,\ A\alpha_n)=k(\eta_1,\eta_2,\ldots,\eta_s)A\\
&=(\eta_1,\eta_2,\ldots,\eta_s)(kA),
\end{aligned}$$

□□□□□ kA□□□□ kA. □□

$$\sigma(kA)=kA=k\sigma(A), \tag{9}$$

□□□σ□□□□□□□□.

□□□, σ□□□□□ $\mathrm{Hom}(V,V')$□ $M_{s\times n}(F)$□□□□□□□, □□□□□□□□□□:

□□ 1 □ V□V'□□□□ F□ n□□s□□□□□, □

$$\mathrm{Hom}(V,V')\cong M_{s\times n}(F) \tag{10}$$

□□

$$\dim\mathrm{Hom}(V,V')=\dim M_{s\times n}(F)=sn=(\dim V)(\dim V'). \tag{11}$$

□□ 2 □ V□□ F□□ n□□□□□, □

$$\mathrm{Hom}(V,V)\cong M_n(F), \tag{12}$$

□

$$\dim\mathrm{Hom}(V,V)=(\dim V)^2. \tag{13}$$

□ $\mathrm{Hom}(V,V)$□ $M_n(F)$□, □□□□□□. □□□□□□□□□: □□□□□ A□□□□□ V□□ $\alpha_1,\alpha_2,\ldots,\alpha_n$ □□□□ A□□□σ□□□□□□□□: □□□□□ B□ V□□ $\alpha_1,\alpha_2,\ldots,\alpha_n$ □□□□□ $B=(b_{ij})$. □□

$$
\begin{aligned}
&(\mathscr{A}\mathscr{B})(\alpha_1,\alpha_2,\ldots,\alpha_n)\\
&=\mathscr{A}(\mathscr{B}\alpha_1,\ \mathscr{B}\alpha_2,\ \ldots,\ \mathscr{B}\alpha_n)\\
&=\mathscr{A}\ ((\alpha_1,\alpha_2,\ldots,\alpha_n)B)\\
&=\mathscr{A}\ (b_{11}\alpha_1+b_{21}\alpha_2+\ldots+b_{n1}\alpha_n,\ \ldots,\ b_{1n}\alpha_1+b_{2n}\alpha_2+\ldots+b_{nn}\alpha_n)\\
&=(b_{11}\mathscr{A}\alpha_1+b_{21}\mathscr{A}\alpha_2+\ldots+b_{n1}\mathscr{A}\alpha_n,\ \ldots,\ b_{1n}\mathscr{A}\alpha_1+b_{2n}\mathscr{A}\alpha_2+\ldots+b_{nn}\mathscr{A}\alpha_n)\\
&=(\mathscr{A}\alpha_1,\ \mathscr{A}\alpha_2,\ldots,\ \mathscr{A}\alpha_n)\begin{pmatrix} b_{11} & \ldots & b_{1n}\\ \ldots & & \ldots\\ b_{n1} & \ldots & b_{nn}\end{pmatrix}\\
&=(\mathscr{A}(\alpha_1,\alpha_2,\ldots,\alpha_n))B\\
&=((\alpha_1,\alpha_2,\ldots,\alpha_n)A)B=(\alpha_1,\alpha_2,\ldots,\alpha_n)(AB),
\end{aligned}
\tag{14}
$$

□□ $\mathscr{A}\mathscr{B}$ □□ $\alpha_1,\alpha_2,\ldots,\alpha_n$ □□□□□ AB. □□

$$\sigma(\mathscr{A}\mathscr{B})=AB=\sigma(\mathscr{A})\sigma(\mathscr{B}), \tag{15}$$

□□□ σ □□□□□□.

□(14)□□□□□□□□□□□,

$$\mathscr{A}\ ((\alpha_1,\alpha_2,\ldots,\alpha_n)B)\ =\ (\mathscr{A}(\alpha_1,\alpha_2,\ldots,\alpha_n))\ B. \tag{16}$$

□□, V□□□□□□ $\mathscr{E}$□□ $\alpha_1,\alpha_2,\ldots,\alpha_n$ □□□□□□□□□□ E, □□ $\sigma(\mathscr{E})=E$.

□ V□□□□□ $\mathscr{A}$□□ $\alpha_1,\alpha_2,\ldots,\alpha_n$□□□□□ A, □□

□□□□ $\mathscr{A}$□□

- □□ V□□□□□□ $\mathscr{B}$□□ $\mathscr{A}\mathscr{B}=\mathscr{B}\mathscr{A}=\mathscr{E}$
- □□ V□□□□□□ $\mathscr{B}$□□ $\sigma(\mathscr{A})\sigma(\mathscr{B})=\sigma(\mathscr{B})\sigma(\mathscr{A})=\sigma(\mathscr{E})$
- □□□ F□ n□□□ B□□ $AB=BA=E$
- □□ A□□,

□□, V□□□□□ $\mathscr{A}$□□□□□□□ V□□□□□□□□ A□□. □□□□□□□□□□□, □□□□□ $\mathscr{A}$, $\mathscr{B}$□ V□□□□□□□□□□□ A, B, □ B □□□□□□□ $\mathscr{A}$□□□□□□□□ $\mathscr{B}$□□□□□ $\mathscr{A}$□□□□.

□ $\mathscr{A}$□□ F□ n□□□□□ V□□□□□□□, □ $\mathscr{A}$□ V□□□□ $\alpha_1,\alpha_2,\ldots,\alpha_n$ □□□□□ A. V□□□□□ α □□ $\alpha_1,\alpha_2,\ldots,\alpha_n$□□□□□ X, □: $\mathscr{A}\alpha$ □□ $\alpha_1,\alpha_2,\ldots,\alpha_n$□□□□□□□ ?

□□ $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n)X$, □□□□(16)□, □□

$$
\begin{aligned}
\mathscr{A}\alpha &= \mathscr{A}\ ((\alpha_1,\alpha_2,\ldots,\alpha_n)X)\ =\ (\mathscr{A}(\alpha_1,\alpha_2,\ldots,\alpha_n))\ X\\
&=((\alpha_1,\alpha_2,\ldots,\alpha_n)A)X=(\alpha_1,\alpha_2,\ldots,\alpha_n)(AX).
\end{aligned}
$$

□□□ $\mathscr{A}\alpha$ □□ $\alpha_1,\alpha_2,\ldots,\alpha_n$ □□□□□ AX.

□□ V□□□□□□□□□□□□□□ V□□□□□□□□□,□□,□□□□ γ □□ $\alpha_1,\alpha_2,\ldots,\alpha_n$□□□□□ Y,□

$$\mathcal{A}\alpha=\gamma \Leftrightarrow AX=Y. \tag{17}$$

□ F□ n□□□□□ V□□□□□□□ $\mathcal{A}$□ V□□□□□□□□□□□□□□?

□□ **3** □ V□□ F□ n□□□□□, V□□□□□□□□ $\mathcal{A}$□ V□□□□ $\alpha_1,\alpha_2,\ldots,\alpha_n$□ $\eta_1,\eta_2,\ldots,\eta_n$□□□□□□□ A, B. □□ $\alpha_1,\alpha_2,\ldots,\alpha_n$□□ $\eta_1,\eta_2,\ldots,\eta_n$□□□□□□ S,□

$$B=S^{-1}AS. \tag{18}$$

□□ □□□□□,□□□

$$\mathcal{A}(\alpha_1,\alpha_2,\ldots,\alpha_n)=(\alpha_1,\alpha_2,\ldots,\alpha_n)A,$$
$$\mathcal{A}(\eta_1,\eta_2,\ldots,\eta_n)=(\eta_1,\eta_2,\ldots,\eta_n)B,$$
$$(\eta_1,\eta_2,\ldots,\eta_n)=(\alpha_1,\alpha_2,\ldots,\alpha_n)S,$$

□□ $$(\eta_1,\eta_2,\ldots,\eta_n)S^{-1}=(\alpha_1,\alpha_2,\ldots,\alpha_n)SS^{-1}=(\alpha_1,\alpha_2,\ldots,\alpha_n)(SS^{-1})=(\alpha_1,\alpha_2,\ldots,\alpha_n),$$

□□

$$\begin{aligned}
&\mathcal{A}(\eta_1,\eta_2,\ldots,\eta_n)\\
&=\mathcal{A}(\alpha_1,\alpha_2,\ldots,\alpha_n)S\\
&=\mathcal{A}(\alpha_1,\alpha_2,\ldots,\alpha_n)S=(\alpha_1,\alpha_2,\ldots,\alpha_n)AS\\
&=(\alpha_1,\alpha_2,\ldots,\alpha_n)(AS)=(\eta_1,\eta_2,\ldots,\eta_n)S^{-1}(AS)\\
&=(\eta_1,\eta_2,\ldots,\eta_n)S^{-1}AS.
\end{aligned} \tag{19}$$

(19)□□□, $\mathcal{A}$□□ $\eta_1,\eta_2,\ldots,\eta_n$□□□□□ $S^{-1}AS$. □□ $\mathcal{A}$□□ $\eta_1,\eta_2,\ldots,\eta_n$□□□□□□□□,□□ $B=S^{-1}AS$. x

□□ 3□□,□□□□□□□ $\mathcal{A}$□ V□□□□□□□□□□□□.

□ □ 9 3

1 □ $\mathcal{A}$□ K^3□□□□□□□□:

$$\mathcal{A}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}x_1+2x_2\\x_3-x_2\\x_2-x_3\end{pmatrix},$$

□ $\mathcal{A}$□□□□ $\varepsilon_1,\varepsilon_2,\varepsilon_3$□□□□.

2 □ R^R□,□□□□□□□

$$f_1 = e^{ax}\cos bx,\ f_2 = e^{ax}\sin bx$$

□□□ 2□□□□□□ V,□□□□□ D□ V□□□□□□□□,□□□ D□ V□□□□ f_1, f_2□□□□.

3 $\mathscr{A}$□ $M_2(K)$□□□□□□□□:

$$\mathscr{A}(X) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} X,\quad \forall X \in M_2(K),$$

□ $\mathscr{A}$□□ $E_{11}, E_{12}, E_{21}, E_{22}$□□□□.

4 $\mathscr{B}$□ $M_2(K)$□□□□□□□□:

$$\mathscr{B}(X) = X\begin{pmatrix} a & b \\ c & d \end{pmatrix},\quad \forall X \in M_2(K),$$

□ $\mathscr{B}$□□ $E_{11}, E_{12}, E_{21}, E_{22}$□□□□.

5 □ V□□ F□□ n□□□□□,□□ V□□□□□□ $\mathscr{A}$□ V□□□□ α,□□ $\mathscr{A}^{n-1}\alpha \neq 0$,□ $\mathscr{A}^n\alpha = 0$.□□:V□□□□□□,□□ $\mathscr{A}$□□□□□□□□□□

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

6 □ $\mathscr{A}$□□ F□ n□□□□□ V□□□□□□□□□,$\mathscr{A}$□ V□□□□□□□□□□ A.□□:$\mathscr{A}$□□□□□□□□□ A□□□□□.

7. □ $\mathscr{A}$□□ F□ n□□□□□ V□□□□□□□□.□□:

(1) □ $F[x]$□□□□□□□□ n^2□□□□□□ $f(x)$□□ $f(\mathscr{A}) = 0$;

(2) □□ $f(\mathscr{A}) = 0$, $g(\mathscr{A}) = 0$,□ $d(\mathscr{A}) = 0$,□□ $d(x)$□ $f(x)$□ $g(x)$□□□□□□□;

(3) $\mathscr{A}$□□□□□□□□□□:□□□□□□□□□□□□□ $f(x)$□□ $f(\mathscr{A}) = 0$.

8 □ V□□ F□□ n□□□□□,$\alpha_1, \ldots, \alpha_n$□ V□□□□.□ V□□□□□□□□ $\mathscr{A}$□□ $\alpha_1, \ldots, \alpha_n$□□□□□ A,□□:

$$\operatorname{rank} \mathscr{A} = \operatorname{rank} A$$

* 9 □ V□ V'□□□ F□□□□□□□□,$\mathscr{A}$□ V□ V'□□□□□□□.□□:□□ V□□□□□ V'□□□□,□□ $\mathscr{A}$□□□□□□□□□ A□□

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

10 □□ K^3□□□□□□ $\mathscr{A}$□□□□ $\varepsilon_1, \varepsilon_2, \varepsilon_3$□□□□□

$$A = \begin{pmatrix} 15 & -11 & 5 \\ 20 & -15 & 8 \\ 8 & -7 & 6 \end{pmatrix},$$

□ $\eta_1 = (2, 3, 1)'$,$\eta_2 = (3, 4, 1)'$,$\eta_3 = (1, 2, 2)'$,□ $\mathscr{A}$□□ η_1, η_2, η_3□□□□ B.

11 □□ K^3□□□□□ $\mathscr{A}$□□ $\alpha_1 = (8, -6, 7)'$,$\alpha_2 = (-16, 7, -13)'$,$\alpha_3 = (9, -3, 7)'$□

□□□□□

$$A=\begin{pmatrix}1&-18&15\\-1&-22&20\\1&-25&22\end{pmatrix},$$

□ A□□ $\eta_1=(1,-2,1)$□, $\eta_2=(3,-1,2)$□, $\eta_3=(2,1,2)$□□□□□ B.

12 □□ F□ 3□□□□□ V□□□□□□ A□□ $\alpha_1,\alpha_2,\alpha_3$□□□□□

$$A=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix},$$

(1) □ A□□ $\alpha_2,\alpha_3,\alpha_1$□□□□;

(2) □ A□□ $k\alpha_1,\alpha_2,\alpha_3$□□□□,□□ k□ F□ k□ 0;

(3) □ A□□ $\alpha_1,\alpha_1+\alpha_2,\alpha_3$□□□□.

*13 □ A□□ F□ n□□□□□ V□□□□□□□□.□□:□□ A□ V□□□□□□□□□□□□□,□ A□□□□□.

14 □ $\alpha_1,\alpha_2,\alpha_3,\alpha_4$□□□ K□ 4□□□□□ V□□□□,□□□□□□ A□□□□□□□□□

$$A=\begin{pmatrix}1&0&2&1\\-1&2&1&3\\1&2&5&5\\2&-2&1&-2\end{pmatrix},$$

(1) □ A□□ $\eta_1=\alpha_1-2\alpha_2+\alpha_4,\ \eta_2=3\alpha_2-\alpha_3-\alpha_4,\ \eta_3=\alpha_3+\alpha_4,\ \eta_4=2\alpha_4$□□□□;

(2) □ A□□□□□;

(3) □ Ker A□□□□□,□□□□□ V□□□□,□□ A□□□□□□□□;

(4) □ Im A□□□□□,□□□□□ V□□□□,□□□ A□□□□□□□□.

§4 □□□□□□□□□□□□□

□□□□□ 3□□,□ F□ n□□□□□ V□□□□□□ A□ V□□□□□□□□□□□□.□□□□□□□□□□□□□□□□□□□□□□□,□□□□□□□□□□□ A□ V□□□□□□□□ A□□□□□□□□□□□□□□□□□□□□□□□□□ A□□□□□□□□□□□□□□□□(□§2□,□□□□ A□□□□□ Im A□□□,□□□□ A□ V□□□□□□□□ A□□□□ A□□,□□□□□□□ 9 3□□ 8□□□□ dim Im A =rank A).

□□□□□□□□□□□□□□□□□□□□,□□□□□□□□□□□□□□□□□□□□□□□,□□□□□□□□□:

□□1 □ A□□ F□□□□□ V□□□□□□□□,□□ V□□□□□□□□□ξ,□□

$$\mathcal{A}\xi=\lambda_0\xi,\quad \lambda_0 □ F \tag{1}$$

□□λ_0□ $\mathcal{A}$□□□□□□,□ξ□ $\mathcal{A}$□□□□□λ_0□□□□□□□.

□□□1□□,□□□□ $\mathcal{A}$□□□□□ξ□□□□“□□□□”:$\mathcal{A}$□ξ□□□□□ξ“□□(□□□)”λ_0□,□□□□ λ_0□□ $\mathcal{A}$□□□□□□.

□□□ V□□ F□ n□□□□□,V□□□□□□$\alpha_1,\alpha_2,\dots,\alpha_n$. V□□□□□□□ $\mathcal{A}$□□$\alpha_1,\alpha_2,\dots,\alpha_n$□□□□□ A,□□ξ□□$\alpha_1,\alpha_2,\dots,\alpha_n$□□□□□ X,$\lambda_0$□ F.□§3□(17)□□□

$$\mathcal{A}\xi=\lambda_0\xi \iff AX=\lambda_0X. \tag{2}$$

□□□□

$$\lambda_0□ \mathcal{A}□□□□□□ \iff \lambda_0□ A□□□□□□; \tag{3}$$

ξ□ $\mathcal{A}$□□□□□□λ_0□□□□□□

$$\iff ξ□□□ X□ A□□□□□□\lambda_0□□□□□□□. \tag{4}$$

(3)□□□,□□□□□□□□□,□□□□□□□□□□□□□□□□□□□□□□,□□□1□□□□.

(3)□□(4)□□□□□□□□□□□□□□□□□□ $\mathcal{A}$□□□□□□□□□□□□□:□□□□ $\mathcal{A}$□ V□□□□□□□□ A□□□□□□□□□□.□□□□□:□□ A□□□□□ X□□□□□ $\mathcal{A}$□□□□□ξ□□□.

□1 □ V□□□ K□3□□□□□,$\mathcal{A}$□ V□□□□□□□,$\mathcal{A}$□ V□□□□$\alpha_1,\alpha_2,\alpha_3$□□□□□

$$A=\begin{pmatrix}2&-2&2\\-2&-1&4\\2&4&-1\end{pmatrix},$$

□ $\mathcal{A}$□□□□□□□□□□□.

□ A□□□□□□□

$$|\lambda I-A|=\begin{vmatrix}\lambda-2&2&-2\\2&\lambda+1&-4\\-2&-4&\lambda+1\end{vmatrix}=(\lambda-3)^2(\lambda+6),$$

□□ A□□□□□□□ 3(□□),-6.

□□□□ 3,□□□□□□□□□(3I-A)X=0,□□□□□□□□:

$$\begin{pmatrix}-2\\1\\0\end{pmatrix},\quad\begin{pmatrix}2\\0\\1\end{pmatrix}.$$

□ $$\xi_1=-2\alpha_1+\alpha_2,\ \xi_2=2\alpha_1+\alpha_3,$$

□ A□□□□□ 3□□□□□□□

$$\{k_1\xi_1+k_2\xi_2 \mid k_1,k_2 \,□\, K, □\ k_1,k_2 □□□\ 0\}.$$

□□□□□ -6,□□ $(-6I-A)X=0$ □□□□□□□:

$$\begin{pmatrix}1\\2\\-2\end{pmatrix}.$$

□ $$\xi_3=\alpha_1+2\alpha_2-2\alpha_3,$$

□ A□□□□□ -6 □□□□□□□

$$\{k\xi_3 \mid k \,□\, K, □\ k □ 0\}.$$

□ A□□ F□□□□□ V□□□□□□□□,λ_0□ A□□□□□□.□

$$V_{\lambda_0} \overset{\text{def}}{=} \{\alpha \mid A\alpha=\lambda_0\alpha, \alpha □ V\}, \tag{5}$$

□□□□ V_{λ_0}□ V□□□□□□,□ V_{λ_0}□ A□□□□□□λ_0□□□□□□. V_{λ_0}□□□□□□□□□□ A□□□λ_0□□□□□□.

□ V□□ F□ n□□□□□, V□□□□□ A□ V□□□□$\alpha_1,\alpha_2,\ldots,\alpha_n$□□□□□ A,$\lambda_0$□ A□□□□□□,□□(5)□□(2)□□□,□□□□ A□□□λ_0□□□□□□ V_{λ_0}□□□□□□□□□□□□□□□□ A□□□λ_0□□□□□□,□□□□□□□□□□□□$(\lambda_0 I-A)X=0$□□□□,□□

$$\dim V_{\lambda_0}=n-\operatorname{rank}(\lambda_0 I-A). \tag{6}$$

□□□□□□,□□□□□□□,□□□□□□□□□ A,□ V□□□□□□□□□□,□□ A□□□□□□□□□□□□□□□□.□□ A□ V□□□□□□□□□□□□,□□□□□□□□□□ A□ V□□□□□□□□ A□□□□□□.□□□□□□□□□□ n□□□ A□□□□□□□□,□□□ A□□□□□□□□□□□□($□$ A□□□□)□□□□□□□.□□□□□□□□□□□□ A,□□□□□□□□□□?□□□□□□□□.□□□□□,□□□□□□□ A,□□□□□ V□□□□□□□,□□ A□□□□□□□□□□□□□□□.

□□ V□□□□□□,□□□□□□ A□□□□□□□□□□□□□□,□□ A□□□□.

□□□□□ A□ V□□□□$\alpha_1,\alpha_2,\ldots,\alpha_n$□□□□□ A,□ A□□□□□□□□ A□□□□.□□□ n□□□ A□□□□□□□□□□□□(□□ 5□ §6□□□ 1□□ 5,□ 8□□□ 8.2□□ 16□),□□ V□ F^n□□□□□,□□□□□□□ A□□□□□□□□□□□□□□:

□□ 1 □ A□□ F□ n□□□□□ V□□□□□□□□,□

A□□□□

� A□ n□□□□□□□□□□

� V□□□□ A□□□□□□□□□□□

� A□□□□□□□□□□□□□□□□□□□□□ n

� $V = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_s}$,

□□ $\lambda_1, \lambda_2, \ldots, \lambda_s$□ A□□□□□□□□□. X

□□ A□□□□,□ A□ n□□□□□□□□□□ $\xi_1, \xi_2, \ldots, \xi_n$,□□ $A\xi_i = \lambda_i \xi_i$, $i = 1, 2, \ldots, n$. □□

$$A(\xi_1, \xi_2, \ldots, \xi_n) = (\xi_1, \xi_2, \ldots, \xi_n) \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & & \cdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix}. \tag{7}$$

□(7)□□□,□□□□□□□□□□□□□□□□ A□□□□□(□□□□□□□).□□□□□□□□□□□□□□□□□□□□□,□□□□□ A□□□□,□□□□□□□□□□□□□□□ A□□□□.

□ 2 □ 1□□□□□□ A□□□□□□?□□ A□□□□,□ A□□□□.

□ □□ 1□□□□ A□□□□□ 3□□□□□□□□□□□□□ ξ_1, ξ_2, A□□□ -6□□□□□□□ ξ_3. □□□□ 5□§6□□,□□ A□□□□□□□□□□□□□□□□□□□.□□□□□□□□□□□□,ξ_1, ξ_2, ξ_3□□□□□□□.□□ A□□□□.

□□ A□□□□□□□ 3(□□), -6,□□ A□□□□□

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -6 \end{pmatrix}.$$

□□□□□□□□□□□□□ A,□□□□ V□□□□□□□□□,□□ A□□□□□□□□□□□□□□□□□?□□□□□□□□□□□□□□□,□□□□□□□□□□□?□□□□□□□□□□□□□□.

□ □ 9 4

1 □ V□□□ K□ 3□□□□□,V□□□□□□□□ A□ V□□□□ $\alpha_1, \alpha_2, \alpha_3$□□□□□ A,□ A□□□□□□□□□□□□.

(1) $A=\begin{pmatrix}2&2&-2\\2&5&-4\\-2&-4&5\end{pmatrix}$; (2) $A=\begin{pmatrix}2&3&2\\1&8&2\\-2&-14&-3\end{pmatrix}$.

2 □1□□□□□□□ A□□□□□□?□□ A□□□□,□ A□□□□.

3 □ $\alpha_1,\alpha_2,\alpha_3,\alpha_4$□□□ K□4□□□□□ V□□□□,V□□□□□□A□□□□□□□□□

$$A=\begin{pmatrix}1&0&0&0\\0&0&0&0\\1&0&0&0\\0&0&0&1\end{pmatrix}.$$

(1)□ A□□□□□□□□□□□□;

(2)□ V□□□□,□□ A□□□□□□□□□□□□□,□□□□□□□□□□.

4 □ V□□ F□□□□□□□□□(□□□□□□□),A□ V□□□□□□□□.□□:A□□□□□□□□□□□□□□□□□□□□.

5 □ V□□ F□□□□□□□□□,λ_1,λ_2□□□□□ A□□□□□□□□□,ξ_1,ξ_2□□□□□ λ_1,λ_2□□□□□.□□:$\xi_1+\xi_2$□□ A□□□□□.

6 □ V□□ F□□□□□□□,□□:□□ V□□□□□ A□ V□□□□□□□□□□□□□□□,□ A□□□□□.

7. □ V□□ F□□□□□□□,□ A□ V□□□□□□□□□□.

(1)□□ A□□□□□□□□□0;

(2)□□:□□ λ□ A□□□□,□ $\frac{1}{\lambda}$□ A^{-1}□□□□.

8 □ A□□ F□ n□□□□□ V□□□□□□□□,$\lambda_1,\lambda_2,\dots,\lambda_s$□ A□□□□□□□□□.□□:

$$A□□□□ \Leftrightarrow V=V_{\lambda_1}\oplus V_{\lambda_2}\oplus\dots\oplus V_{\lambda_s}.$$

*9 □ A□□□□□ n□□□□□ V□□□□□□□□,A□□□□□□□□□ n□□□($\lambda_1,\lambda_2,\dots,\lambda_n$)□□ A□□.□□ A□□□□□□□□1□□,□ A□□□□□□.□□:□□ A□□□□□,□□□ A□□□□□□□□□□□ B□□□□□□□ A□□□□□□□ n□□□□.

§5 □□□□□□□□□□

□□□□□□□□□□□□□□□□□□,□□□□□□□□□□□,□□□□□□□?□□□□□□□□□□□□□?□§4□□□1□□,A□□□□□□□□□□□□□□□□□□ V□□□□ A□□□□□□□□□.□□□□□□,□□□□□□□□□□□□□□□□,□□□□□□□□ V□□□□ A□□

□□□□□□□□□□□□□□□?□□□□□□□□□□.□□ A □□□□□□ V_{λ_i} □□□□□□:□ α □ V_{λ_i},□ $A\alpha=\lambda_i\alpha_i$ □ V_{λ_i} □□□□□□□ A □□□□□□□□□□.

□□ 1 □ A □□ F □□□□□ V □□□□□, W □ V □□□□.□□ W □□□□□ A □□□□□ W □,□□□□□ α □ W,□□ $A\alpha$ □ W,□□ W □ A □□□□□□,□□ A-□□□.

□□,□□□□ V □□□□□ 0,□□ V □□□□□□□□ A □□,□□ A-□□□.□ V □ 0 □ A □□□□□□□□□.

□□ 1 V □□□□□ A □□□□,A □□□□□□□□ A-□□□.

□□ □□ α □ $\mathrm{Ker}A$,□□ $A\alpha=0$ □ $\mathrm{Ker}A$,□□ $\mathrm{Ker}A$ □ A-□□□.

□□ α □ $\mathrm{Im}A$,□□ $A\alpha$ □ $\mathrm{Im}A$,□□ $\mathrm{Im}A$ □ A-□□□.

□□ α □ V_{λ_i},□□ $A\alpha=\lambda_i\alpha$ □ V_{λ_i},□□ V_{λ_i} □ A-□□□. x

□□ 2 □□□□□□ A □ B □□□(□ $AB=BA$),□ $\mathrm{Ker}B$, $\mathrm{Im}B$, B □□□□□□□□ A-□□□.

□□ □□ α □ $\mathrm{Ker}B$,□ $B\alpha=0$.□□

$$B(A\alpha)=(BA)\alpha=(AB)\alpha=A(B\alpha)=A(0)=0.$$

□□ $A\alpha$ □ $\mathrm{Ker}B$,□□ $\mathrm{Ker}B$ □ A-□□□.

□□□□□ $\mathrm{Im}B$ □ A-□□□.

□ B □□□□□□ V_{λ_i} □□□□□□□□ α,□ $B\alpha=\lambda_i\alpha$.□□

$$B(A\alpha)=A(B\alpha)=A(\lambda_i\alpha)=\lambda_i(A\alpha),$$

□□ $A\alpha$ □ V_{λ_i},□□ V_{λ_i} □ A-□□□. x

□□ 3 □ A □□ F □□□□□ V □□□□□□,□□□□□ $f(x)$ □ $F[x]$,□□ $\mathrm{Ker}f(A)$, $\mathrm{Im}f(A)$, $f(A)$ □□□□□□□□ A-□□□. x

□□,□□□□ A □□□□□□□□□□□□ A □□□□□□.

□□ 4 □ A □□ F □□□□□ V □□□□□□, $W=$□$\alpha_1,\alpha_2,\ldots,\alpha_m$□□ V □□□□□□,□ W □ A □□□□□□□□□□□ $A\alpha_i$ □ W, $i=1,2,\ldots,m$

□□ □□ W □□□□□ α □□□□ $k_1\alpha_1+k_2\alpha_2+\ldots+k_m\alpha_m$,□□,

W □ A-□□□

□ □ α □ W □□□ $A\alpha$ □ W

□ $A(k_1\alpha_1+k_2\alpha_2+\ldots+k_m\alpha_m)$ □ W, " k_i □ F, $i=1,2,\ldots,m$

□ $k_i\sum_{i=1}^{m}A\alpha_i$ □ W, " k_i □ F, $i=1,2,\ldots,m$

□ $A\alpha_i$ □ W, $i=1,2,\ldots,m$ x

□ A □□ F □□□□□ V □□□□□□,□□ W □ A-□□□,□ A □

□□□□□ W□,□□□□□□□,□□□ W□□□□□□□□,□□□ $\mathcal{A}$ □ W□□□□,□□ $\mathcal{A}|W$.□□□□□□

$$(\mathcal{A}|W)\alpha = \mathcal{A}\alpha, \qquad \forall \alpha \in W. \tag{1}$$

□□ W□□ $\mathcal{A}$-□□□,□□□□□□ $\mathcal{A}$□ W□□□□,□□ $\mathcal{A}|W$,□□□ W□□□□□□,□□ W□ V□□□□□□□.

□ $\mathcal{A}$□□ F□□□□□□□□ V□□□□□□,$\mathcal{A}$□□□□□□□□□□□□ $\mathcal{A}$□□□□□□□□□□□□□.

□□ **5** □ $\mathcal{A}$□□ F□ n□□□□□ V□□□□□□,□ $\mathcal{A}$□□□□□□□□□□□□□□ V□□□□□□,□□ $\mathcal{A}$□□□□□□□□□□□□□□□□□□:

$$\begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix}. \tag{2}$$

□□ □□□.□ W□ $\mathcal{A}$□□□□□□□□□,□ W□□□□□ $\alpha_1,\alpha_2,\ldots,\alpha_r$,□□□□□ V□□□□:

$$\alpha_1,\alpha_2,\ldots,\alpha_r,\alpha_{r+1},\ldots,\alpha_n,$$

□ $\qquad \mathcal{A}(\alpha_1,\ldots,\alpha_r,\alpha_{r+1},\ldots,\alpha_n)$

$$=(\alpha_1,\ldots,\alpha_r,\alpha_{r+1},\ldots,\alpha_n)\begin{pmatrix} a_{11} & \ldots & a_{1r} & a_{1,r+1} & \ldots & a_{1n} \\ \ldots & & \ldots & \ldots & & \ldots \\ a_{r1} & \ldots & a_{rr} & a_{r,r+1} & \ldots & a_{rn} \\ 0 & \ldots & 0 & a_{r+1,r+1} & \ldots & a_{r+1,n} \\ \ldots & & \ldots & \ldots & & \ldots \\ 0 & \ldots & 0 & a_{n,r+1} & \ldots & a_{nn} \end{pmatrix}. \tag{3}$$

□□ $\mathcal{A}$□ V□□□□□□□□□

$$\begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix},$$

□□ A_1 □ $\mathcal{A}|W$□ W□□□□ $\alpha_1,\ldots,\alpha_r$□□□□.

□□□.□ $\mathcal{A}$□ V□□□□ $\alpha_1,\alpha_2,\ldots,\alpha_n$□□□□□□□□□□□(2),□□ A_1 □ r□□□,$0<r<n$.□

$$W=\langle\alpha_1,\alpha_2,\ldots,\alpha_r\rangle. \tag{4}$$

□ $\mathcal{A}\alpha_i \in W$, $i=1,2,\ldots,r$.□□ W□ $\mathcal{A}$-□□□,□□ W□□□□□.□□ A_1 □ $\mathcal{A}|W$□ W□□ $\alpha_1,\alpha_2,\ldots,\alpha_r$□□□□. x

□□ **6** □ $\mathcal{A}$□□ F□ n□□□□□ V□□□□□□,□ V□□□□ $\mathcal{A}$□□□□□□□□□□□□□□□□□□ V□□□□□□,□□ $\mathcal{A}$□□□□□□□□□□□□□:

$$\begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_s \end{pmatrix}. \tag{5}$$

□□ □□□.□ V□ A□□□□□□□□□□□□□□:

$$V=W_1\oplus W_2\oplus\cdots\oplus W_s. \tag{6}$$

□□□ $W_i(i=1,2,\dots,s)$□□□□□ $\alpha_{i1},\dots,\alpha_{ir_i}$,□(6)□□□,

$$\alpha_{11},\dots,\alpha_{1r_1},\dots,\alpha_{s1},\dots,\alpha_{sr_i} \tag{7}$$

□ V□□□□.□□ W_i□ A-□□□,□□

$$A(\alpha_{i1},\dots,\alpha_{ir_i})=(\alpha_{i1},\dots,\alpha_{ir_i})A_i,\quad i=1,2,\dots,s. \tag{8}$$

□□ A□(7)□□□□□□□□□□

$$\begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_s \end{pmatrix}. \tag{9}$$

□□□.□ A□ V□□□□

$$\alpha_{11},\dots,\alpha_{1r_1},\dots,\alpha_{s1},\dots,\alpha_{sr_i}$$

□□□□ $A=\mathrm{diag}(A_1,A_2,\dots,A_s)$,□□ A_i□ r_i□□□,$i=1,2,\dots,s$.□ $W_i=$□$\alpha_{i1},\dots,\alpha_{ir_i}$□,$i=1,2,\dots,s$.□□

$$A(\alpha_{i1},\dots,\alpha_{ir_i})=(\alpha_{i1},\dots,\alpha_{ir_i})A_i,\quad i=1,2,\dots,s, \tag{10}$$

□□ $A\alpha_{i1},\dots,A\alpha_{ir_i}$□ W_i.□□ W_i□ A-□□□,□□ W_i□□□□□,□□ W_i□□□□$\alpha_{i1},\dots,\alpha_{ir_i}$□ $i=1,2,\dots,s$□,□□□□ V□□□□,□□ $W_1+W_2+\dots+W_s$□□□,□ $V=W_1+W_2+\dots+W_s$,□□ $V=W_1\oplus W_2\oplus\dots\oplus W_s$. x

□□□6□□□□□□□,(5)□□□□□□□,A_i□□ $A|W_i$□ W_i□□□□$\alpha_{i1},\dots,\alpha_{ir_i}$□□□□,□□ $i=1,2,\dots,s$.

□□□6□□,□□ n□□□□□ V□□□□□□□□ A□□□□□□□□□□□□□□□,□□ V□□□□□□,□□ A□□□□□□□□□□□□□□,□□□□□□□□□.□□□□□□□ A□□□□□□□□□□□?□□□ 3,□□□□ $f(x)$□ $F[x]$,□□ $\mathrm{Ker}\,f(A)$□ A□□□□□□.□□□□□□,□□□□□□□□□□ $f_1(x),f_2(x),\dots,f_s(x)$□ $F[x]$,□□

$$V=\mathrm{Ker}\,f_1(A)\oplus\mathrm{Ker}\,f_2(A)\oplus\cdots\oplus\mathrm{Ker}\,f_s(A). \tag{11}$$

□□□□□□□□□,□□□□□□□□□□ $f_1(x), f_2(x)$, x□ A□□□□□ $f_1(A)$□ $f_2(A)$□□□□□□□□□ ?

□□ 7 □ A□□ F□□□□□ V□□□□□□,□ $F[x]$□, $f(x) = f_1(x)f_2(x)$, □ $(f_1(x), f_2(x)) = 1$, □

$$\mathrm{Ker}\, f(A) = \mathrm{Ker}\, f_1(A) \oplus \mathrm{Ker}\, f_2(A). \tag{12}$$

□□ □□ $\mathrm{Ker}\, f(A) = \mathrm{Ker}\, f_1(A) + \mathrm{Ker}\, f_2(A)$.

□□ $f(x) = f_1(x)f_2(x)$, □□ $f(A) = f_1(A)f_2(A)$. □□ $\alpha_1 \in \mathrm{Ker}\, f_1(A)$, □ $f_1(A)\alpha_1 = 0$. □□□

$$f(A)\alpha_1 = (f_1(A)f_2(A))\alpha_1 = f_2(A)(f_1(A)\alpha_1) = 0,$$

□□ $\alpha_1 \in \mathrm{Ker}\, f(A)$. □□ $\mathrm{Ker}\, f_1(A) \subseteq \mathrm{Ker}\, f(A)$. □□□□, $\mathrm{Ker}\, f_2(A) \subseteq \mathrm{Ker}\, f(A)$. □□ $\mathrm{Ker}\, f_1(A) + \mathrm{Ker}\, f_2(A) \subseteq \mathrm{Ker}\, f(A)$.

□□ $(f_1(x), f_2(x)) = 1$, □□□□ $u_1(x), u_2(x) \in F[x]$, □□ $u_1(x)f_1(x) + u_2(x)f_2(x) = 1$. x□ A□□,□

$$u_1(A)f_1(A) + u_2(A)f_2(A) = I. \tag{13}$$

□□ $\alpha \in \mathrm{Ker}\, f(A)$, □□(13)□□

$$\alpha = I\alpha = u_1(A)f_1(A)\alpha + u_2(A)f_2(A)\alpha. \tag{14}$$

□ $\alpha_1 = u_2(A)f_2(A)\alpha, \alpha_2 = u_1(A)f_1(A)\alpha$.

□□

$$f_1(A)\alpha_1 = f_1(A)u_2(A)f_2(A)\alpha = u_2(A)f(A)\alpha = 0,$$

□□, $\alpha_1 \in \mathrm{Ker}\, f_1(A)$, □□□□, $\alpha_2 \in \mathrm{Ker}\, f_2(A)$. □□□(14)□□

$$\alpha = \alpha_1 + \alpha_2 \in \mathrm{Ker}\, f_1(A) + \mathrm{Ker}\, f_2(A),$$

□□

$$\mathrm{Ker}\, f(A) = \mathrm{Ker}\, f_1(A) + \mathrm{Ker}\, f_2(A).$$

□□ $\mathrm{Ker}\, f_1(A) \cap \mathrm{Ker}\, f_2(A) = 0$. □□ $\beta \in \mathrm{Ker}\, f_1(A) \cap \mathrm{Ker}\, f_2(A)$, □

$$\beta = I\beta = u_1(A)f_1(A)\beta + u_2(A)f_2(A)\beta = 0 + 0 = 0.$$

□□ $\mathrm{Ker}\, f_1(A) \cap \mathrm{Ker}\, f_2(A) = 0$. □□□□,□

$$\mathrm{Ker}\, f(A) = \mathrm{Ker}\, f_1(A) \oplus \mathrm{Ker}\, f_2(A).$$

x

□□□□□□□□□□□ 7□□□:

□□ 8 □ A□□ F□□□□□ V□□□□□□,□ $F[x]$□, $f(x) = f_1(x)f_2(x)\cdots f_s(x)$, □□ $f_1(x), f_2(x), \ldots, f_s(x)$□□□□,□

$$\mathrm{Ker}\, f(A) = \mathrm{Ker}\, f_1(A) \oplus \mathrm{Ker}\, f_2(A) \oplus \cdots \oplus \mathrm{Ker}\, f_s(A). \tag{15}$$

x

□□ $\mathrm{Ker}\, 0 = V$, □□□□□ 8□□□□,□□□□□□□□□□ $f(x)$,□□ $f(A) = 0$, □□□□ V□□□□□

$$V = \mathrm{Ker} f_1(A) \oplus \cdots \oplus \mathrm{Ker} f_s(A), \tag{16}$$

□□, $f_1(x)\cdots f_s(x) = f(x)$, □□ $f_1(x), \cdots, f_s(x)$ □□□□.

□□ 2 □ A□ V□□□□□□, □□ F□□□□□□□ $f(x)$, □□ $f(A) = 0$, □□ $f(x)$□ A□□□□□□□□.

□□□□, □□ $f(x)$□ $F[A]$□□□□□□ A, □□ $f(x)$□ A□□□□□□□□□.

□□ V□□□□□, □□ V□□□□□□□□□ A□□□□□□□□□. □□□□: □ $\dim V = n$, □ $\dim \mathrm{Hom}(V, V) = n^2$. □□ $I, A, A^2, \cdots, A^{n^2}$ □□□□□□. □□□ F□□□□□□□□ $k_0, k_1, \cdots, k_{n^2}$, □□

$$k_0 I + k_1 A + k_2 A^2 + \cdots + k_{n^2} A^{n^2} = 0. \tag{17}$$

□ $$f(x) = k_0 + k_1 x + k_2 x^2 + \cdots + k_{n^2} x^{n^2}$$

□(17)□□, $f(A) = 0$. □ $f(x)$□ A□□□□□□□□□□□.

□□ 3 □ A□□ F□□□ n□□□, □□ $F[x]$□□□□□□ $f(x)$□□ $f(A) = 0$, □□ $f(x)$□□□ A□□□□□□□□.

□ A□□ F□ n□□□□□ V□□□□□□, □ A□ V□□□□ $\alpha_1, \alpha_2, \cdots, \alpha_n$□□□□□ A. □

$$k_0 I + k_1 A + \cdots + k_m A^m = 0 \Longleftrightarrow k_0 I + k_1 A + \cdots + k_m A^m = 0,$$

□□ $k_0, k_1, \cdots, k_m$□ F, m□□□□□. □□

$$f(A) = 0 \Longleftrightarrow f(A) = 0,$$

□ $$f(x)\text{□ } A\text{□□□□□□} \Longleftrightarrow f(x)\text{□ } A\text{□□□□□□}.$$

□ □ 9 5

1 □ K^4□, □□□□□ A□□ $\alpha_1, \alpha_2, \alpha_3, \alpha_4$□□□□□

$$A = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 4 & -2 \\ 2 & -1 & 0 & 1 \\ 2 & -1 & -1 & 2 \end{pmatrix},$$

□ $W=$□$\alpha_1 + 2\alpha_2, \alpha_2 + \alpha_3 + 2\alpha_4$□. □□: W□ A-□□□.

2 □ W□ V□□□□□□□ A□□□□□□□□□, □□:

(1) $A|W$□ W□□□□□□□□;

(2) W□□ A^{-1}□□□□□□, □□ $(A|W)^{-1} = A^{-1}|W$.

3 □ V□□□□□□ n□□□□□, A, B□ V□□□□□□, □□ $AB = BA$. □□: A

□ B□□□□□□□□□□□□.

4 □ V□□ F□□ n□□□□□,□□□□ A□□ $\alpha_1,\alpha_2,\dots,\alpha_n$ □□□□□

$$A=\begin{pmatrix} a & 1 & 0 & \dots & 0 & 0\\ 0 & a & 1 & \dots & 0 & 0\\ \dots & \dots & \dots & & \dots & \dots\\ 0 & 0 & 0 & \dots & a & 1\\ 0 & 0 & 0 & \dots & 0 & a \end{pmatrix},$$

(1) □□:□□ α_n □□ A□□□□□□ W,□ W=V;

(2) □□:α_1□□ A□□□□□□□□□□□□;

(3) □□: V□□□□□ A□□□□□□□□□□□□□□□;

*(4) □ A□□□□□□□□.

5 □ V□□□□□ n□□□□□,□□: V□□□□□□□ A□□□□ 1□□□□□□□□ 2□□□□□□.

6 □ V□□□□□□□ O□□□□□□□□□□□□□□□□□ 2□□□□□, A□□ O□□□□ θ □□□,□□ $\theta\ne k\pi$, $k\in Z$ □□: A□□□□□□□□□□□□□.

*7 □ A□□ F□ n□□□□□ V□□□□□□, λ_0□ A□□□□□□. λ_0□□ A□□□□□□□□□□□□□□ λ_0□□□□□; A□□□ λ_0□□□□□□ V_{λ_0}□□□□□ λ_0□□□□□. □□:λ_0□□□□□□□□□□□□□□.

*8 □ A□□□□□ n□□□□□ V□□□□□□. □□: A□□□□□□□□□□□□□, A□□□□□□□□□□□□□□□□□□□□.

9 □ A□□ F□ n□□□□□ V□□□□□□,□□: A□□□□□□□□□□□□□□

$$\operatorname{rank}(A)+\operatorname{rank}(A-I)=n.$$

*10 □ A□□ F□□ n□□□.□□: A□□□□□(□ $A^2=I$)□□□□□□□□

$$\operatorname{rank}(A+I)+\operatorname{rank}(A-I)=n.$$

11 □□□□□□ k□□□□□□□□□□□□.

12 □□:□□□□□□ A□□□ 1□□□□□□□,□ A□□□□□□□.

13 □□□ 2□□□ A□□□□□□□□□□□□:

$$A=\begin{pmatrix} 1 & 1\\ 0 & 1 \end{pmatrix}.$$

14 □□□ n□□□ A□□□□□□□□□□□□□:

$$A=\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0\\ 0 & 0 & 1 & \dots & 0 & 0\\ \dots & \dots & \dots & & \dots & \dots\\ 0 & 0 & 0 & \dots & 0 & 1\\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

*15 □ W□ V□□□□□ A□□□□□□□□□,□

$$\overline{\mathcal{A}}: V/W \to V/W$$
$$\alpha + W \longmapsto \mathcal{A}\alpha + W,$$

□□:$\overline{\mathcal{A}}$□□□□ V/W□□□□□□□□.

§6 Hamilton-Cayley □□

□□□ K□□2□□□ A□

$$A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix},$$

□

$$A^2 = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

□ $A^2 - I = 0$. □□ $f(\lambda) = \lambda^2 - 1$□ A□□□□□□□□.□□

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ 0 & \lambda + 1 \end{vmatrix} = (\lambda - 1)(\lambda + 1) = \lambda^2 - 1.$$

□□ A□□□□□□ $f(\lambda) = \lambda^2 - 1$□□ A□□□□□□□□.□□□□□□□ F□□□□□ n□□□ A□□□?□□□□□□□□,□□□□□□□□□□□□□□□□□□□□□.

□ R□□□□□,R□ n^2□□□□□□ n□ n□□□□□□□ R□□□□ n□□□.□□□□ F□□□□□□□□□□□□□□□□□□,□□□□□□□□ R□□□□□□□□□□□□□□□,□□□□□□□□□□ F□□□□□□□□□□.□□ F□ n□□□□□□□□□□□□□□,□□□□□□ R□□ n□□□□□□□,□□□□□□□□□2□□□□□□□□□□□□□□□□(□)□□□□□,□□□□ R□□ n□□□□□□□□□□,□□□□□□ R□□□□ A□□

$$AA^* = A^*A = |A|I,$$

□□ A^*□ A□□□□□.

□ F□□□□□□□□□ $F[\lambda]$□□□□□,□□□□□□□ $F[\lambda]$□□ n□□□,□□□□□□□ λ-□□,□□ $A(\lambda), B(\lambda), \ldots,$□.□□

$$A(\lambda) = \begin{pmatrix} 2\lambda^3 + \lambda^2 + 1 & \lambda^2 - 3 \\ \lambda^3 - 1 & 2\lambda + 5 \end{pmatrix} \tag{1}$$

□□□ λ-□□,□□□□□□□□□□□□□□□□,□□□(1)□□ λ-□□ $A(\lambda)$□□

$$A(\lambda) = \begin{pmatrix} 2\lambda^3 & 0 \\ \lambda^3 & 0 \end{pmatrix} + \begin{pmatrix} \lambda^2 & \lambda^2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 2\lambda \end{pmatrix} + \begin{pmatrix} 1 & -3 \\ -1 & 5 \end{pmatrix}$$

$$=\lambda^3\begin{pmatrix}2&0\\1&0\end{pmatrix}+\lambda^2\begin{pmatrix}1&1\\0&0\end{pmatrix}+\lambda\begin{pmatrix}0&0\\0&2\end{pmatrix}+\begin{pmatrix}1&-3\\-1&5\end{pmatrix}, \qquad (2)$$

□□λ^i□“□□□□”□□ F□□□□.

□□□□□□λ-□□ $A(\lambda)$□ $B(\lambda)$□□□(2)□□□□□□,□□□□□□□□□□□□□□□□□λ-□□□□□□□,□□□□, $A(\lambda)$□ $B(\lambda)$□□□□□□□□□λ^i□“□□□□”□□□□, $i=0,1,2,\ldots$.

□□□□□□□□□,□□□□□□□□ Hamilton-Cayloy□□.

Hamilton-Cayley□□ □ A□□ F□□ n□□□,□ A□□□□□□ $f(\lambda)$□ A□□□□□□□□.

□□ □ $B(\lambda)$□ λ-□□$(\lambda I - A)$□□□□□,□

$$B(\lambda)(\lambda I - A) = |\lambda I - A| I = f(\lambda) I. \qquad (3)$$

□□ $B(\lambda)$□□□□□□ $\lambda I - A$□□□(□□□□□□□□□)□□□□□□,□□ $B(\lambda)$□□□□□□□□□□ $n-1$□□□□□□.□□ $B(\lambda)$□□□□

$$B(\lambda) = \lambda^{n-1}B_{n-1} + \lambda^{n-2}B_{n-2} + \ldots + \lambda B_1 + B_0, \qquad (4)$$

□□ $B_{n-1}, B_{n-2}, \ldots, B_1, B_0$□□□ F□□ n□□□.

□□□□□

$$\begin{aligned} &B(\lambda)(\lambda I - A)\\ &=(\lambda^{n-1}B_{n-1} + \lambda^{n-2}B_{n-2} + \ldots + \lambda B_1 + B_0)(\lambda I - A) \qquad (5)\\ &=\lambda^n B_{n-1} + \lambda^{n-1}(B_{n-2} - B_{n-1}A) + \ldots + \lambda(B_0 - B_1 A) - B_0 A. \end{aligned}$$

□□ A□□□□□□ $f(\lambda)$□□□□ n,□□□□□

$$f(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0. \qquad (6)$$

□□
$$f(\lambda)I = \lambda^n I + a_{n-1}\lambda^{n-1}I + \ldots + a_1\lambda I + a_0 I. \qquad (7)$$

□(3),(5),(7)□□

$$\begin{aligned} &B_{n-1} = I,\\ &B_{n-2} - B_{n-1}A = a_{n-1}I,\\ &\cdots\cdots\cdots\cdots \qquad (8)\\ &B_0 - B_1 A = a_1 I,\\ &-B_0 A = a_0 I. \end{aligned}$$

□ $A^n, A^{n-1}, \ldots, A, I$□□□□□□(8)□□1□,□2□,…,□ n□,□ $n+1$□,□

$$\begin{aligned} &B_{n-1}A^n = A^n,\\ &B_{n-2}A^{n-1} - B_{n-1}A^n = a_{n-1}A^{n-1},\\ &\cdots\cdots\cdots\cdots \qquad (9)\\ &B_0 A - B_1 A^2 = a_1 A,\\ &-B_0 A = a_0 I, \end{aligned}$$

□(9)□ $n+1$□□□□□□□□,□□□□,□□□□ $f(A)$.□□ $f(A)=0$. x

□□□□□□□□□□□□□□,□

Hamilton-Cayley□□ □ A□□ F□ n□□□□□ V□□□□□□□□,□ A□□□□□□ $f(\lambda)$□ A□□□□□□□□. x

□□ 9 6

1 □ A□□ F□ n□□□□□ V□□□□□□.

□□:□□ A□□□□□ $f(\lambda)$□ $F[\lambda]$□□□□□

$$f(\lambda)=(\lambda-\lambda_1)^{r_1}(\lambda-\lambda_2)^{r_2}\cdots(\lambda-\lambda_s)^{r_s},$$

□

$$V=\mathrm{Ker}(A-\lambda_1 I)^{r_1}\oplus\mathrm{Ker}(A-\lambda_2 I)^{r_2}\oplus\cdots\oplus\mathrm{Ker}(A-\lambda_s I)^{r_s},$$

□□ $\mathrm{Ker}(A-\lambda_j I)^{r_j}$□□ A□□□□□, $j=1,2,\ldots,s$.

2 □□:□□□ F□□ n□□□□□ A,□□ F□□□ $k_0,k_1,\ldots,k_{n-1}$,□□

$$A^{-1}=k_{n-1}A^{n-1}+\cdots+k_1A+k_0I.$$

3 □□□□□□□□□□□□□□□□:

(1) $\begin{pmatrix}1&2\\-1&4\end{pmatrix}$; (2) $\begin{pmatrix}1&-1\\1&1\end{pmatrix}$.

*4 □ A,B□□□ n□□m□□□□,□□:□□□□ $AX-XB=0$□□□□□□□□□□□,A□ B□□□□□□□.

§7 □□□□□□□□□□

□ Hamilton-Cayley□□□□,□□□□□□□ V□□□□□□ A□□□□□□ $f(\lambda)$□ A□□□□□□□□.□□□□□□□□ A□□□□□□□□□□?□□□□□□,□□ A□□□□□□□□□□□□□?

□□ 1 □ A□□ F□□□□□ V□□□□□□□,□ A□□□□□□□□□□□□,□□□□□□□□□□ 1□□□□□□ A□□□□□□.

□□ 1 □□□□ V□□□□□□ A□□□□□□□□□□.

□□ □ $m_1(\lambda)$□ $m_2(\lambda)$□□ A□□□□□□,□□□□□□□□□□□□□□□□ 1,□□ $h(\lambda)=m_1(\lambda)-m_2(\lambda)$□□□□ $m_1(\lambda)$□□.□□ $h(A)=m_1(A)-m_2(A)=0$,□□ $h(\lambda)$□□ A□□□□□□□□.□□□□□□□□□□, $h(\lambda)=0$.□

$$m_1(\lambda)=m_2(\lambda).$$ x

□□2 □□□□ V□□□□□□ $\mathcal{A}$□□□□□□□□ $g(\lambda)$□ $\mathcal{A}$□□□□□□ $m(\lambda)$□□□.

□□ □□□□□,□

$$g(\lambda)=h(\lambda)m(\lambda)+r(\lambda),\quad \deg r(\lambda)<\deg m(\lambda).$$

□□□λ□ $\mathcal{A}$□□,□□□□

$$g(\mathcal{A})=h(\mathcal{A})m(\mathcal{A})+r(\mathcal{A}).$$

□□ $g(\mathcal{A})=0$, $m(\mathcal{A})=0$, □□ $r(\mathcal{A})=0$. □□□□□□□□□□,$r(\lambda)=0$. □□ $g(\lambda)$□ $m(\lambda)$□□□. ×

□□3 □ $\mathcal{A}$□□ F□□□□□□□□ V□□□□□□,□ $\mathcal{A}$□□□□□□ $m(\lambda)$□□□□□□ $f(\lambda)$□ F□□□□□□(□□□□□□).

□□ □ Hamilton-Cayley□□□□□ 2□,$f(\lambda)=h(\lambda)m(\lambda)$,□□□□ $h(\lambda)$□ $F[\lambda]$. □□ $m(\lambda)$□□□□□□ $f(\lambda)$□□.

□□,□ λ_0□ $f(\lambda)$□ F□□□□□,□ λ_0□ $\mathcal{A}$□□□□□□.□□□□ ξ□ V□ξ□0,□□ $\mathcal{A}\xi=\lambda_0\xi$.□

$$m(\lambda)=c_0+c_1\lambda+\dots+c_s\lambda^s,$$

□

$$\begin{aligned}0=m(\mathcal{A})\xi&=(c_0I+c_1\mathcal{A}+\dots+c_s\mathcal{A}^s)\xi\\&=c_0\xi+c_1\lambda_0\xi+\dots+c_s\lambda_0^s\xi=m(\lambda_0)\xi.\end{aligned}$$

□□□□,$m(\lambda_0)=0$,□ λ_0□ $m(\lambda)$□□□□. ×

□□□,□□□□□ F□ n□□□ A□□□□□□.

□ n□□□□□ V□□□□□□ $\mathcal{A}$□ V□□□□□□□□□ A,□§5□□□□□□,$g(\lambda)$□ $\mathcal{A}$□□□□□□□□□ $g(\lambda)$□ A□□□□□□.□□□□,$m(\lambda)$□ $\mathcal{A}$□□□□□□□□□□ $m(\lambda)$□ A□□□□□□.□□□□□3□□,

□□4 □ F□ n□□□ A□□□□□□ $m(\lambda)$□□□□□□ $f(x)$□ F□□□□□□(□□□□□□). ×

□□□□□□□□□□□□□□□□□□□ $\mathcal{A}$□ V□□□□□□□□,□□□□□□□:

□□5 □□□□□□□□□□□□□□□. ×

□□□□□□□ $\mathcal{A}$(□ n□□□ A)□□□□□□?□□□□□:□□□ $\mathcal{A}$□□□□□□□□ $g(\lambda)$,□□□□ $g(\lambda)$□□□□□□ $\mathcal{A}$□□□□□□.

□□2 □ $\mathcal{A}$□□ F□□□□□ V□□□□□□,□□□□□□□□□□ l,□□ $\mathcal{A}^l=0$,□□ $\mathcal{A}$□□□□□;□□ $\mathcal{A}^l=0$□□□□□□□□ l□□ $\mathcal{A}$□□□□□.

□1 □□□□□□ l□□□□□ $\mathcal{A}$□□□□□□.

□ □□ $A^l=0$, □□ $g(\lambda)=\lambda^l$ □□□□□ A □□□□□□□. □□□□□□□□□, □ $0<s<l$ □, A^s □ 0. □□ λ^l □ A □□□□□□.

□ **2** □□ F □□ r □□□ A □

$$A=\begin{pmatrix}0&1&0&\cdots&0&0\\0&0&1&\cdots&0&0\\\cdots&\cdots&\cdots&&\cdots&\cdots\\0&0&0&\cdots&0&1\\0&0&0&\cdots&0&0\end{pmatrix},$$

□ A □□□□□□.

□ □ W □□ F □ r □□□□□, V □□□□□ $\alpha_1,\alpha_2,\ldots,\alpha_r$, □□ W □□□□□□□ A, □□ $A(\alpha_1,\alpha_2,\ldots,\alpha_r)=(\alpha_1,\alpha_2,\ldots,\alpha_r)A$.

□ $$A\alpha_1=0,\ A\alpha_2=\alpha_1,\ A\alpha_3=\alpha_2,\ \ldots,\ A\alpha_r=\alpha_{r-1}.$$

□□ $$A^2\alpha_2=A(A\alpha_2)=A\alpha_1=0,$$
$$A^3\alpha_3=A^2(A\alpha_3)=A^2(\alpha_2)=0,$$
$$\cdots\cdots\cdots\cdots$$
$$A^r\alpha_r=A^{r-1}(A\alpha_r)=A^{r-1}(\alpha_{r-1})=0.$$

□□ A □□□□□, □□□□□□ r. □□ 1 □□, A □□□□□□□ λ^r. □□□□ A □□□□□□□ λ^r.

□□ 2 □□□□□□□, □ 2 □□□ A □□□□□, □□□□□□□□□□□□□□, □□□□□□□□□□□, $A^r=0$; □ $0<s<r$ □, A^s □ 0.

□□ 2 □□□□□□□□□□□□□:

□□ **3** □ F □□□□ r □□□□□□□

$$\begin{pmatrix}a&1&0&\cdots&0&0\\0&a&1&\cdots&0&0\\\cdots&\cdots&\cdots&&\cdots&\cdots\\0&0&0&\cdots&a&1\\0&0&0&\cdots&0&a\end{pmatrix},\qquad(1)$$

□□□□□□ r □ Jordan □, □□ $J_r(a)$, □□ a □□□□□□□□□.

□ 2 □□□□□□□□□ 0 □ r □ Jordan □ $J_r(0)$.

1 □ Jordan □ $J_1(a)$ □□ 1 □□□ (a).

□ **3** □ r □ Jordan □ $J_r(a)$ □□□□□□.

□ □□ $J_r(a)$ □□□□□□ $f(\lambda)$:

$$f(\lambda)=|\lambda I-J_r(a)|=\begin{vmatrix}\lambda-a & -1 & 0 & \cdots & 0 & 0\\ 0 & \lambda-a & -1 & \cdots & 0 & 0\\ \cdots & \cdots & \cdots & & \cdots & \cdots\\ 0 & 0 & 0 & \cdots & \lambda-a & -1\\ 0 & 0 & 0 & \cdots & 0 & \lambda-a\end{vmatrix}=(\lambda-a)^r.$$

□□ $J_r(a)$□□□□□□ $m(\lambda)$□ $f(\lambda)$□□□,□□

$$m(\lambda)=(\lambda-a)^s, \text{□□□□ } s(0<s\le r).$$

λ□ $J_r(a)$□□,□

$$0=m(J_r(a))=(J_r(a)-aI)^s=J_r(0)^s.$$

□□ $J_r(0)$□□□□□□ r,□□ $s=r$.□□ $J_r(a)$□□□□□□ $m(\lambda)=(\lambda-a)^r$.

□□□□□6□□□□□□□ A□□□□□□□□□□□.

□□ **6** □ A□□ F□□□□□ V□□□□□□.□□ V□□□□ A□□□□□□□□□□□□□□:

$$V=W_1\oplus W_2\oplus\cdots\oplus W_s, \tag{2}$$

□ A□□□□□□ $m(\lambda)$□

$$m(\lambda)=[m_1(\lambda),m_2(\lambda),\dots,m_s(\lambda)], \tag{3}$$

□□ $m_j(\lambda)$□ W_j□□□□□□ $A|W_j$□□□□□□, $j=1,2,\dots,s$. $[m_1(\lambda),m_2(\lambda),\dots,m_s(\lambda)]$□ $m_1(\lambda),m_2(\lambda),\dots,m_s(\lambda)$□□□□□□.

□□ □ $g(\lambda)=[m_1(\lambda),m_2(\lambda),\dots,m_s(\lambda)]$,□

$$g(\lambda)=h_j(\lambda)m_j(\lambda),\quad j=1,2,\dots,s.$$

□□ $g(\lambda)|m(\lambda)$,□□□□□ $m_j(\lambda)|m(\lambda)$, $j=1,2,\dots,s$.□□ $\alpha_j\in W_j$,□

$$m(A|W_j)\alpha_j=m(A)\alpha_j=0\alpha_j=0,$$

□□ $m(A|W_j)=0$,□ $m(\lambda)$□ W_j□□□□□ $A|W_j$□□□□□□□□.□□ $m_j(\lambda)|m(\lambda)$.□□ $j=1,2,\dots,s$,□□ $m(\lambda)$□ $m_1(\lambda),m_2(\lambda),\dots,m_s(\lambda)$□□□□,□□□□□□□□□□ $g(\lambda)|m(\lambda)$.

□□ $m(\lambda)|g(\lambda)$,□□□□□ $g(\lambda)$□ A□□□□□□.□□ $\alpha\in V$.□□ $V=W_1\oplus W_2\oplus\cdots\oplus W_s$,□□

$$\alpha=\alpha_1+\alpha_2+\cdots+\alpha_s,\quad \alpha_i\in W_i,\ i=1,2,\dots,s.$$

□□

$$g(A)\alpha=g(A)\sum_{j=1}^{s}\alpha_j=\sum_{j=1}^{s}g(A)\alpha_j=\sum_{j=1}^{s}h_j(A)m_j(A)\alpha_j$$

$$=\sum_{j=1}^{s} h_j(A)\quad m_j(A|\ W_j)\alpha_j\quad =\sum_{j=1}^{s} h_j(A)[0\alpha_j]\ =0.$$

□□ $g(A)=0$. □□ $g(\lambda)$□ A□□□□□□. □□ $m(\lambda)\,|\,g(\lambda)$.

□□□□□, $m(\lambda)=g(\lambda)=\ m_1(\lambda),\ m_2(\lambda),\dots,\ m_s(\lambda)$. x

□□ **7** □ A□□ F□□□ n□□□□□□□, □ $A=\mathrm{diag}\{A_1, A_2,\dots, A_s\}$, □ A_j□□□□□□□ $m_j(\lambda)$, □ A□□□□□□ $m(\lambda)$□

$$m(\lambda)=\ m_1(\lambda),\ m_2(\lambda),\dots,\ m_s(\lambda)\ .$$

□□ □ V□□ F□ n□□□□□, V□□□□□ $\alpha_1,\alpha_2,\dots,\alpha_n$, □□ V □□□□□□□□ A, □□

$$A(\alpha_1,\alpha_2,\dots,\alpha_n)=(\alpha_1,\alpha_2,\dots,\alpha_n)A.$$

□□ A□□□□□□□, □ §5□□□ 6□, V□□□□ A□□□□□□□□ □□□□□□:

$$V=W_1\oplus W_2\oplus\cdots\oplus W_s,$$

□□ A_j□□ $A|W_j$□ W_j□□□□□□□□. □□ $A|W_j$□□□□□□□□ A_j □□□□□□ $m_j(\lambda)$, $j=1,2,\dots,s$. □ A□□□□□□□□ A□□□□□□ $m(\lambda)$. □□□ 6□

$$m(\lambda)=\ m_1(\lambda),\ m_2(\lambda),\dots,\ m_s(\lambda)\ .$$ x

□□ **4** □□□□ Jordan□□□□□□□□□□□□ Jordan□□□.

□□□□□□□□□□ 1□ Jordan□□□□ Jordan□□□.

□ **4** □ A□ Jordan□□□:

$$A=\mathrm{diag}\ J_3(a), J_5(a), J_2(b)\ ,$$

□ A□□□□□□.

□ $J_3(a), J_5(a)$□□□□□□□□□ $(\lambda-a)^3,(\lambda-a)^5$, $J_2(b)$□□□□ □□□ $(\lambda-b)^2$. □□ A□□□□□□□

$$m(\lambda)=\ (\lambda-a)^3,(\lambda-a)^5,(\lambda-b)^2$$
$$=(\lambda-a)^5(\lambda-b)^2.$$

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□.□ □□□□□□□□□□□□□□□□□□□□□□□□□□□□.□□□□□ □□□□□□□□□□□□□□□□□□□.

□□ **8** □ A□□ F□ n□□□□□ V□□□□□□, □ A□□□□□ □□□□□□□, A□□□□□□ $m(\lambda)$□ $F[\lambda]$□□□□□□□□□□□□□ □□.

□□ □□□. □ A□□□□, □

$$V=V_{\lambda_1}\oplus V_{\lambda_2}\oplus\cdots\oplus V_{\lambda_s},$$

□□ $\lambda_1,\lambda_2,\ldots,\lambda_s$□ $\mathcal{A}$□□□□□□□□□. □□ α_j□ V_{λ_j}, □ $\mathcal{A}\alpha_j=\lambda_j\alpha_j$, □□ $(\mathcal{A}-\lambda_j I)\alpha_j=0$, □□ $\mathcal{A}|V_{\lambda_j}-\lambda_j I\ \alpha_j=0$. □□ $\mathcal{A}|V_{\lambda_j}-\lambda_j I=0$, □□ $\lambda-\lambda_j$□ $\mathcal{A}|V_{\lambda_j}$□□□□□□□□, □□ $\mathcal{A}|V_{\lambda_j}$□□□□□□ $m_j(\lambda)$□ $\lambda-\lambda_j$□□□, □□ $m_j(\lambda)=\lambda-\lambda_j$. □□ $\mathcal{A}$□□□□□□ $m(\lambda)$□

$$m(\lambda)=\lambda-\lambda_1,\lambda-\lambda_2,\ldots,\lambda-\lambda_s$$
$$=(\lambda-\lambda_1)(\lambda-\lambda_2)\ldots(\lambda-\lambda_s).$$

□□□, □ $\mathcal{A}$□□□□□□ $m(\lambda)$□

$$m(\lambda)=(\lambda-\lambda_1)(\lambda-\lambda_2)\ldots(\lambda-\lambda_s),\qquad (4)$$

□□ $\lambda_1,\lambda_2,\ldots,\lambda_s$□ F□□□□□□□□. □§5□□□ 8□,

$$\mathrm{Ker}\ m(\mathcal{A})=\mathrm{Ker}(\mathcal{A}-\lambda_1 I)\oplus\mathrm{Ker}(\mathcal{A}-\lambda_2 I)\oplus\ldots\oplus\mathrm{Ker}(\mathcal{A}-\lambda_s I).$$

□□, $\mathrm{Ker}\ m(\mathcal{A})=\mathrm{Ker}\,0=V$. □□

$$\alpha\in\mathrm{Ker}(\mathcal{A}-\lambda_j I)\Leftrightarrow(\mathcal{A}-\lambda_j I)\alpha=0\Leftrightarrow\alpha\in V_{\lambda_j},$$

□□ $\mathrm{Ker}(\mathcal{A}-\lambda_j I)=V_{\lambda_j}$, $j=1,2,\ldots,s$. □□

$$V=V_{\lambda_1}\oplus V_{\lambda_2}\oplus\ldots\oplus V_{\lambda_s},$$

□□ $\mathcal{A}$□□□□. x

□□ 9 □ F□ n□□□ A□□□□□□□□□□□□, A□□□□□□ F[λ]□□□□□□□□□□□□□□□. x

□ 5 □□: □□□□□□□□□□.

□□ □ $\mathcal{A}$□□□□□, □ $\mathcal{A}^2=\mathcal{A}$, □□ $\mathcal{A}^2-\mathcal{A}=0$. □□ $\lambda^2-\lambda$□ $\mathcal{A}$□□□□□□□□, □□

$$\lambda^2-\lambda=\lambda(\lambda-1).$$

□□ $\mathcal{A}$□□□□□□ $m(\lambda)$□ $\lambda^2-\lambda$□□□, □□ $m(\lambda)$□□□□□□□□□□□□□□□, □□ $\mathcal{A}$□□□□. x

□ 6 □□: □□□□ $l>1$□□□□□□□□□□□□□.

□□ □ $\mathcal{A}$□□□□□ $l>1$□□□□□, □ $\mathcal{A}$□□□□□□ $m(\lambda)=\lambda^l$. □□ $l>1$, □□ $\mathcal{A}$□□□□□. x

□□ 5□□ 6□□, □□□□□□, □□□□□□□□□□□□□□□□□□□□□□□, □□□□□.

□□□□□□□□□□□□□□□□□□□□□□□□, □□□□□□□□ n□□□□□ V□□□□, □□ V□□□□□□ $\mathcal{A}$□□□□□□□ $\mathcal{A}$□□□ ?

□ $\mathcal{A}$□□ F□ n□□□□□ V□□□□□□, □□ $\mathcal{A}$□□□□□□ $m(\lambda)$□ F[λ]□□□□□□□□□□□□□:

$$m(\lambda)=(\lambda-\lambda_1)^{l_1}(\lambda-\lambda_2)^{l_2}\cdots(\lambda-\lambda_s)^{l_s}, \tag{5}$$

□□ $\lambda_1,\lambda_2,\ldots,\lambda_s$ □ F □□□□□□□□. □

$$\mathrm{Ker}\, m(A)=\mathrm{Ker}(A-\lambda_1 I)^{l_1}\oplus\mathrm{Ker}(A-\lambda_2 I)^{l_2}\oplus\cdots\oplus\mathrm{Ker}(A-\lambda_s I)^{l_s}. \tag{6}$$

□□, $\mathrm{Ker}\, m(A)=\mathrm{Ker}\,0=V$. □ $W_j=\mathrm{Ker}(A-\lambda_j I)^{l_j}$, □ §5 □□□ 3 □, W_j □ A □□□□□□, $j=1,2,\ldots,s$. □(6)□□,

$$V=W_1\oplus W_2\oplus\cdots\oplus W_s. \tag{7}$$

□ W_j □□□□□, $j=1,2,\ldots,s$, □□□□□□□ V □□□□, A □ V □□□□□□□□ A □□□□□□□:

$$A=\begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_s \end{pmatrix},$$

□□ A_j □ $A|W_j$ □ W_j □□□□□□□□. □□□□□ A □□□, □□□□ A_j □□□, $j=1,2,\ldots,s$. □□□□□□□ W_j □□□□□□ $A|W_j$. □□□□ $A|W_j$ □□□□□□□ $(\lambda-\lambda_j)^{l_j}$, □□□□:

□□ $\alpha_j\in W_j$, □□ $W_j=\mathrm{Ker}(A-\lambda_j I)^{l_j}$, □□

$$(A|W_j-\lambda_j I)^{l_j}\alpha_j=(A-\lambda_j I)^{l_j}\alpha_j=0,$$

□□ $(A|W_j-\lambda_j I)^{l_j}=0$, □□ $(\lambda-\lambda_j)^{l_j}$ □ $A|W_j$ □□□□□□□□, □□ $A|W_j$ □□□□□□ $m_j(\lambda)=(\lambda-\lambda_j)^{t_j}$, □□ $t_j\le l_j$. □□ A □□□□□□ $m(\lambda)$ □

$$\begin{aligned} m(\lambda)&=[(\lambda-\lambda_1)^{t_1},(\lambda-\lambda_2)^{t_2},\ldots,(\lambda-\lambda_s)^{t_s}]\\ &=(\lambda-\lambda_1)^{t_1}(\lambda-\lambda_2)^{t_2}\cdots(\lambda-\lambda_s)^{t_s}. \end{aligned} \tag{8}$$

□□□□□□□□□, □(8)□□(5)□□

$$t_1=l_1,\ t_2=l_2,\ \ldots,\ t_s=l_s.$$

□□ $A|W_j$ □□□□□□ $m_j(\lambda)=(\lambda-\lambda_j)^{l_j}$.

□ $B_j=A|W_j-\lambda_j I$, $j=1,2,\ldots,s$. □□ $A|W_j$ □□□□□□ $m_j(\lambda)=(\lambda-\lambda_j)^{l_j}$, □□

$$(A|W_j-\lambda_j I)^{l_j}=0,\quad (A|W_j-\lambda_j I)^{k_j}\ne 0,\ □\ 0\le k_j<l_j.$$

□□□□, B_j □ W_j □□□□□□, □□□□□□ l_j. □□ $A|W_j$ □ W_j □□□□□□□□□ A_j, □□ $B_j=A|W_j-\lambda_j I$ □ W_j □□□□□□□□ $B_j=A_j-\lambda_j I$. □□□□ A_j □□□, □□□□ B_j □□□. □□□□□□□□□:

□ B □□ F □ r □□□□□ W □□□□□□, □□□□□□ l, □□□ W □□□□□□, □□ B □□□□□□□□□□?

□□□□□□□□□□□□□.

□ □ 9 7

1 □□□□□□□□□□□□:

(1) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$; (2) $\begin{pmatrix} 3 & 1 & -1 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$.

2 □□□□□ K□ Jordan□□□□□□□□□:

(1) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; (2) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$;

(3) $\begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$; (4) $\begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$.

3 □ 2□□□(3)□(4)□□□□□□□□□ ?

4 □ 1□□□□□□□□□□□□□□ ?

5 □ 2□□□□ K□□□□ Jordan□□□□□□□□□ ?

6 □□:□□□□□□□□□□□□□□□.

7. □□□ K□□ n□□□ A□□

$$A^3 = 3A^2 + A - 3I,$$

□□ A□□□□□□.

8 □□□ K□□ n□□□ A□□

$$A^3 = A^2 + 4A - 4I,$$

□□ A□□□□□□.

9 □ A□□□□□ Q□□ n□□□□□,□ A□□□□□□□□ $g(\lambda)$□ Q□ r□□□□□□□, $r>1$,□□ A□□□□□□.□□□ A□□□□□,□□ A□□□□□□ ?

10 □ V□□□ K□ 3□□□□□, V□□□□□□ $\mathcal{A}$□□$\alpha_1,\alpha_2,\alpha_3$□□□□ A□

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}.$$

(1) □ $\mathcal{A}$□□□□□□ $m(\lambda)$;

(2) □□□ $m(\lambda)$□□□□□,□□ V□□□□□□,□□□□□□□□□□□□□□□□□□□□.

11 □ $\mathcal{A}$□□ F□ n□□□□□ V□□□□□□, $\mathcal{A}$□ V□□□□ $\alpha_1,\alpha_2,\ldots,\alpha_n$□□□□ A□

$$A=\begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \cdots & \cdots & & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix},$$

□ A□□□□□ $m(\lambda)$□□□□□ $f(x)$.

*12 □ A□□ F□□□□□ V□□□□□,□□:□□ A□□□□□ $m(\lambda)$□□□,□□ $F[A]$□□□□.

*13 □□:□□□ F□□ n□□□ A□ B□□□□□□□,□□ $AB=BA$,□□□□ F□□□ n□□□□□ S,□□ $S^{-1}AS$□ $S^{-1}BS$□□□□□.

§8 □□□□□□□

□□□□□□□,□□□□,□□□□□□□□□□□ A□□□□□,□□ A□□□□□ $m(\lambda)$□ $F[\lambda]$□□□□□□□□□□□,□□□□□□□□□□□□□□□.□□□□ F□ r□□□□□ W□□□□□□ B,□□□ W□□□□□□,□□ B□□□□□□□□□□?

□□ 1 □ B□□ F□ r□□□□□ W□□□□□□,□□□□□□ l,□□□ ξ□ W,□□□□□

$$B^{l-1}\xi,\ B^{l-2}\xi,\ \dots,\ B\xi,\ \xi$$

□□□□,□□□□□□ l□□□□□ $\dim W$.

□□ □□ B□□□□□□ l,□□ B^{l-1}□ 0.□□□□ ξ□ W,□□ $B^{l-1}\xi$□ 0.□□ $B^{l-2}\xi,\dots,B\xi,\xi$□□□□ 0,□ $B^{l}\xi=0\xi=0$.□□□ 9 1□ 9□□□□□,$B^{l-1}\xi,B^{l-2}\xi,\dots,B\xi,\xi$□□□□.□□ l□ $\dim W$. x

□□□□□□□□ $l=\dim W$□□□.□□□□□ 1□,□□ ξ□ W,□□ $B^{l-1}\xi,B^{l-2}\xi,\dots,B\xi,\xi$□□□□.□□ $l=\dim W$,□□□□□□ W□□□□.□□

$$B(B^{l-1}\xi)=B^{l}\xi=0,\ B(B^{l-2}\xi)=B^{l-1}\xi,\ \dots,\ B(\xi)=B\xi,$$

□□ B□ W□□ $B^{l-1}\xi,B^{l-2}\xi,\dots,B\xi,\xi$□□□□□

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{1}$$

□□(1)□□□□□□ 0 □ r □ Jordan □ $J_r(0)$.

□□□□□□□□ $l<\dim W$ □□□.

□□，□□□□ 1 □，□□ ξ □ W，□□ $B^{l-1}\xi,\dots,B\xi,\xi$ □□□□，□□□□ $l<\dim W$，□□□□□□ W □□□□．□□□□□□□□□□□ $\xi,B\xi,\dots,B^{l-1}\xi$ □□ B □□□□□□，□□ B □□□□□□□□□□□□□ $B^{l-1}\xi,\dots,B\xi,\xi$ □□□□□□□ Jordan □ $J_l(0)$(□，□□□□(1))．□□□□□□□□□□□□□：

□□ 1 □ η □ W，□□□□□□□ t，□□

$$B^{t-1}\eta \,□\, 0,\ □\ B^{t}\eta=0.$$

□□□□□ $\eta, B\eta,\dots,B^{t-1}\eta$ □□□□ η □□□ B-□□□□□，□□ $Z_t(\eta;B)$.

□□，$Z_t(\eta;B)$ □ B □□□□□．□□□ 9 1 □□ 9 □□□□□，$B^{t-1}\eta,\dots,B\eta,\eta$ □□□□，□□□□□ $Z_t(\eta;B)$ □□□□．□□ t □ $Z_t(\eta;B)$ □□□．□□□□□□□□，$B|Z_t(\eta;B)$ □□□□□□□□□□□ t □ Jordan □ $J_t(0)$．□□□□□□□ W □□□□□□ B-□□□□□□□，□□ B □□□□□□□□□ Jordan □□□□□□□□□□，□ Jordan □□□．□□□□□□□□，□□□□□□□□□□ $Z_t(\eta;B)$：

$$\begin{aligned}
&\alpha \,□\, Z_t(\eta;B)\\
□\ &\alpha=k_{t-1}B^{t-1}\eta+\dots+k_1B\eta+k_0\eta\\
&\ \ = k_{t-1}B^{t-1}+\dots+k_1B+k_0I\ \eta\\
□\ &\alpha=h(B)\eta,\quad h(B)=k_{t-1}B^{t-1}+\dots+k_1B+k_0I.
\end{aligned}$$

□□

$$Z_t(\eta;B)=\ h(B)\eta\mid h(\lambda)\,□\,F[\lambda],\deg h(\lambda)<t\ . \tag{2}$$

□□□ $Z_t(\eta;B)$ □□ η □□□□□：□□ $h(\lambda)$ □ $F[\lambda]$，□ $\deg h(\lambda)<t$，η □ $h(B)$ □□□ $h(B)\eta$ □□ $Z_t(\eta;B)$．□□□□□，□□□□□□□□：

□□ 2 □ W □□ F □ r □□□□□，B □ W □□□□□□□□，$\eta_1,\eta_2,\dots,\eta_s$ □ W □□□□□．□□ α □ W，□□□□ $h_1(\lambda),h_2(\lambda),\dots,h_s(\lambda)$ □ $F[\lambda]$，□□

$$\alpha=h_1(B)\eta_1+h_2(B)\eta_2+\dots+h_s(B)\eta_s, \tag{3}$$

□□ α □□□ $\eta_1,\eta_2,\dots,\eta_s$ $F[\lambda]$-□□，□ $h(\lambda)$ □□ η_i □□□□□．□□ V □□□□□ α □□□□ $\eta_1,\eta_2,\dots,\eta_s$ $F[\lambda]$-□□，□□ $\eta_1,\eta_2,\dots,\eta_s$ □ W □□□ B-□□□.

□□，W □□□□□ $\alpha_1,\alpha_2,\dots,\alpha_r$ □□ W □□□ B-□□□，□□□□□□□ α □ W，□

$$\begin{aligned}
\alpha&=k_1\alpha_1+k_2\alpha_2+\dots+k_r\alpha_r\\
&=(k_1I)\alpha_1+(k_2I)\alpha_2+\dots+(k_rI)\alpha_r,
\end{aligned}$$

□□ k_i □ F, □□ k_i □ $F[\lambda]$, $i=1,2,\ldots,r$.

□□ 3 □ B□□ F□ r□□□□□ W□□□□□. □□ W□□□ B-□□□□ $\eta_1,\eta_2,\ldots,\eta_m$□ m□□□, □ W□□□ $m-1$□□□□□□ W□ B-□□□□, □□ $\eta_1,\eta_2,\ldots,\eta_m$□ W□□□□ B-□□□□.

□□□□□□□□□□□□□□□□□□, □□ W□□□□□□ B-□□□□. W□□□ B-□□□□□□□.

□□, □ B□□ F□ r□□□□□ W□□□□□, □□□□□□ l, □□□ ξ□ W, □□ $B^{l-1}\xi$□ 0. □□ ξ□□□ B-□□□□□ $Z_l(\xi;B)$□□□□□□ $B|Z_l(\xi;B)$-□□□□: ξ.

□□□□ B-□□□□, □□□□□□□□:

□□ 2 □ B□□ F□□□□□□□□ W□□□□□□, □ W□□□□□□ B-□□□□□□□□.

□□ □□ W□□□ B-□□□□□□□□□□□ s□□□□□□.

$s=1$□, □ η□ W□□□□□ B-□□□□. □□ W□□□□□□□□, □□□□□□□□□ t, □□ $B^{t-1}\eta$□ 0, □ $B^t\eta=0$. □□ η□ W□ B-□□□□, □□

$$W=\{h(B)\eta \mid h(\lambda)\in F[\lambda], \deg h(\lambda)<t\}$$
$$=Z_t(\eta;B).$$

□□ $s=m-1$□, □□□□, □□ $s=m$□□□. □ W□□□□□ B-□□□□□ $\eta_1,\eta_2,\ldots,\eta_m$. □□

$$h_1(B)\eta_1+h_2(B)\eta_2+\cdots+h_m(B)\eta_m=0 \qquad (4)$$

□□□□□□□□□□□□□□□□. □□ $h_i(\lambda)=0$, $i=1,2,\ldots,m$, □□□□□□□□□□; □□□□□□□□. W□ B-□□□□□□□□□□□□(□□, □ $B^{t-1}\eta_1$□ 0, $B^t\eta_1=0$, □ $B^t\eta_1+0\eta_2+\cdots+0\eta_m=0$, □□ $h_1(\lambda)=\lambda^t$□ 0, □□□□□□□□□□□). □ W□□□□□ B-□□□□□□□□□□, □□□□□□□□□□□□□□□□□□□□□ 1□□□□□□□□□(□□□□□□□□□□□□□□□□), □□ $f_1(\lambda)$. □□□ $f_1(\lambda)$□ W□□□ B-□□□□ $\eta_1,\eta_2,\ldots,\eta_m$□ η_1□□□□□□, □ $\eta_1,\eta_2,\ldots,\eta_m$□□□□□:

$$f_1(B)\eta_1+h_2(B)\eta_2+\cdots+h_m(B)\eta_m=0. \qquad (5)$$

□□□□ $f_1(\lambda)\mid h_2(\lambda)$. □□□□: □

$$h_2(\lambda)=g(\lambda)f_1(\lambda)+r(\lambda), \quad \deg r(\lambda)<\deg f_1(\lambda). \qquad (6)$$

□□(5)□□

$$0=f_1(B)\eta_1+[g(B)f_1(B)+r(B)]\eta_2+\cdots+h_m(B)\eta_m$$
$$=f_1(B)[\eta_1+g(B)\eta_2]+r(B)\eta_2+\cdots+h_m(B)\eta_m. \qquad (7)$$

W α

$$\alpha = p_1(B)\eta_1 + p_2(B)\eta_2 + \cdots + p_m(B)\eta_m$$
$$= p_1(B)[\eta_1 + g(B)\eta_2] + [p_2(B) - p_1(B)g(B)]\eta_2 + \cdots + p_m(B)\eta_m,$$

$\eta_1 + g(B)\eta_2, \eta_2, \ldots, \eta_m$ W B- . $f_1(\lambda)$,(7) , $r(\lambda) = 0$. $f_1(\lambda) \mid h_2(\lambda)$.

$f_1(\lambda) \mid h_j(\lambda), j = 3, \ldots, m$

$$h_j(\lambda) = g_j(\lambda) f_1(\lambda), \quad j = 3, \ldots, m,$$

(5)

$$f_1(B)\eta_1 + g(B) f_1(B)\eta_2 + g_3(B) f_1(B)\eta_3 + \cdots + g_m(B) f_1(B)\eta_m = 0. \quad (8)$$

$$\xi_1 = \eta_1 + g(B)\eta_2 + g_3(B)\eta_3 + \cdots + g_m(B)\eta_m. \quad (9)$$

(9) , η_1 $\xi_1, \eta_2, \ldots, \eta_m$ $F[\lambda]$- , $\xi_1, \eta_2, \ldots, \eta_m$ W B- . (8) ,

$$f_1(B)\xi_1 = 0. \quad (10)$$

$f_1(\lambda)$, $f_1(\lambda)$ $f_1(B)\xi_1 = 0$ 1 . ξ_1 k_1, $B^{k_1-1}\xi_1 \neq 0$, $B^{k_1}\xi_1 = 0$. $f_1(\lambda) = \lambda^{k_1}$. ξ_1 k_1 B- $Z_{k_1}(\xi_1; B)$.

$$U = \{h_2(B)\eta_2 + \cdots + h_m(B)\eta_m \mid h_j(\lambda) \in F[\lambda], j = 2, \ldots, m\}, \quad (11)$$

, U W , U B . B|U U , $\xi_1, \eta_2, \ldots, \eta_m$ W B- ,

$$W = Z_{k_1}(\xi_1; B) + U. \quad (12)$$

$\gamma \in Z_{k_1}(\xi_1; B) \cap U$, $q_i(\lambda) \in F[\lambda], i = 1, 2, \ldots, m$, $\deg q_1(\lambda) < k_1$,

$$\gamma = q_1(B)\xi_1 = q_2(B)\eta_2 + \cdots + q_m(B)\eta_m, \quad (13)$$

$$q_1(B)\xi_1 - q_2(B)\eta_2 - \cdots - q_m(B)\eta_m = 0. \quad (14)$$

$\deg q_1(\lambda) < k_1 = \deg f_1(\lambda)$, $q_1(\lambda) \neq 0$, (14) $f_1(\lambda)$. $q_1(\lambda) = 0$. , $\gamma = q_1(B)\xi_1 = 0\xi_1 = 0$. $Z_{k_1}(\xi_1, B) \cap U = 0$.

$$W = Z_{k_1}(\xi_1; B) \oplus U. \quad (15)$$

$\eta_2, \ldots, \eta_m$ U B- (, U m-1 B- , (12) , W m

B-□□□□,□□). □□□□□, U□□□□□□ B|U-□□□□□□□ □:

$$U = Z_{k_2}(\xi_2; B|U) \oplus \cdots \oplus Z_{k_m}(\xi_m; B|U). \tag{16}$$

□□□ $Z_{k_j}(\xi_j; B|U) = Z_{k_j}(\xi_j; B)$, $j = 2, \dots, m$ □□

$$W = Z_{k_1}(\xi_1; B) \oplus Z_{k_2}(\xi_2; B) \oplus \cdots \oplus Z_{k_m}(\xi_m; B). \tag{17}$$

□□□□□□□□□,□□□□. X

□□□□ 2,□□□□□□□□□□□□□□□□.□□□□□□□□:

□□ 3 □ B□□ F□ r□□□□□ W□□□□□□,□□□□□□ l.□ W□□□□□□,□□ B□□□□□□□ B□Jordan□□□,□□□□Jordan□□□□□□□□ 0,□□□□□□ l,Jordan□□□□□□ B□□□□□□ W_0□□□,t□Jordan□□□□ $N(t)$□

$$N(t) = \operatorname{rank} B^{t+1} + \operatorname{rank} B^{t-1} - 2\operatorname{rank} B^{t}; \tag{18}$$

□□Jordan□□□ B□□□□□□ B□Jordan□□□,□□Jordan□□□□□□□,B□Jordan□□□□□□□.

□□ □□□2□□

$$W = \bigoplus_{i=1}^{m} Z_{k_i}(\xi_i; B). \tag{19}$$

□□ $B^l = 0$,□□ k_i□ l, $i = 1, 2, \dots, m$□ $Z_{k_i}(\xi_i; B)$□□□□□ $B^{k_i-1}\xi_i, \dots, B\xi_i, \xi_i$, $i = 1, 2, \dots, m$,□□□□□□□ W□□□□.B□□□□□□□ B□

$$B = \operatorname{diag}\{J_{k_1}(0), J_{k_2}(0), \dots, J_{k_m}(0)\}, \tag{20}$$

□□Jordan□□□.

□□□□□0□□□Jordan□□□□□□□□□,□□ B□Jordan□□□□ m□□ B□□□□□□,□□□□ r□□ B□□□□□□□□.□ B□□□□□□□□□ $\operatorname{rank} B$,□□ $m = r - \operatorname{rank} B$.□□□□□□□□□□□□□ $BX = 0$□□□□□□□,□□□□□□ B□□□□□0,□□□□□$(0I - B)X = 0$□□□□□□□□□ B□□□□□□ W_0□□□.□□

$$m = \dim W_0. \tag{21}$$

□□ n□Jordan□ $J_n(0)$, $\operatorname{rank} J_n(0) = n - 1$.□□□□□,□ $1 < s < n$□,

$$
\begin{array}{ccccccccccc}
 & & s & & & & & & & & \\
 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & \\
 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & \\
 & \cdots & & \cdots & \cdots & \cdots & \cdots & & \cdots & \cdots & \\
J_n(0)^s = & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & \\
 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \\
 & \cdots & & \cdots & \cdots & \cdots & \cdots & & \cdots & \cdots & s \\
 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 &
\end{array}
\qquad (22)
$$

$1 < s < n$, $\operatorname{rank} J_n(0)^s = n - s$. , s n, $J_n(0)^s = 0$. , Jordan $J_n(0)$ n . t Jordan $N(t)$, $\operatorname{rank} B^t$, 1 t l,

$$B^t = \operatorname{diag}\ J_{k_1}(0)^t,\ J_{k_2}(0)^t,\ \ldots,\ J_{k_m}(0)^t\ . \qquad (23)$$

k_i t, $J_{k_i}(0)^t = 0$. $\operatorname{rank} B^t$, t Jordan t , Jordan t t,

$$
\begin{aligned}
\operatorname{rank} B^t &= (t+1) - t\ N(t+1) + (t+2) - t\ N(t+2) + \ldots + (l - t)\ N(l) \\
&= N(t+1) + 2N(t+2) + \ldots + (l - t)\ N(l). \qquad (24)
\end{aligned}
$$

2 t l, (24)

$$\operatorname{rank} B^{t-1} = N(t) + 2N(t+1) + \ldots + (l - t + 1)\ N(l) \qquad (25)$$

$t = 1$, (25) $= \operatorname{rank} B^0 = \operatorname{rank} I = r$, (25)

$$N(1) + 2N(2) + \ldots + l\,N(l) = r,$$

(25) .

(25) (24)

$$\operatorname{rank} B^{t-1} - \operatorname{rank} B^t = N(t) + N(t+1) + \ldots + N(l). \qquad (26)$$

1 $t < l$, (26) t $t+1$

$$\operatorname{rank} B^t - \operatorname{rank} B^{t+1} = N(t+1) + N(t+2) + \ldots + N(l). \qquad (27)$$

1 $t < l$, (26) (27)

$$\operatorname{rank} B^{t-1} + \operatorname{rank} B^{t+1} - 2\operatorname{rank} B^t = N(t) \qquad (28)$$

$t = l$, (28) $= \operatorname{rank} B^{l-1} = l - (l-1)\ \ N(l) = N(l)$, $= N(l)$, (28) 1 t l . (28)

$$N(t) = \operatorname{rank} B^{t+1} + \operatorname{rank} B^{t-1} - 2\operatorname{rank} B^t.$$

B Jordan 0, Jordan B , t Jordan $N(t)$ B^{t+1}, B^{t-1}, B^t . Jordan , B Jordan . x

3 :

□□4 □ B□□ F□□□□□□,□□□□□□ l.□ B□□□□□ Jordan□□□,□□□□ Jordan□□□□□□□ 0,□□□□□□ l,Jordan□□□□□□ B□□□□□□□□(□□□□□□□ $BX=0$ □□□□□□),t□ Jordan□□□□ $N(t)$□

$$N(t)=\operatorname{rank}B^{t+1}+\operatorname{rank}B^{t-1}-2\operatorname{rank}B^{t};\tag{29}$$

□□ Jordan□□□□ B□ Jordan□□□,□□ Jordan□□□□□□□,B□ Jordan□□□□□□□.

x

□ □ 9 8

1 □□□ K□□ 4□□□ B□

$$\begin{pmatrix} -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

(1) □□ B□□□□□,□ B□□□□□;

(2) □ B□ Jordan□□□.

2 □□:□□□ F□□ n□□□ B□□□□□,□□□□□□□ k,□ $\operatorname{tr}(B^k)=0$.

3 □□:n□□□□ B□□□□□□□□□□ B□□□□□□ 0.

§9 □□□□□ Jordan□□□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□.

□□1 □ A□□ F□ n□□□□□ V□□□□□□,□□ A□□□□□□ $m(\lambda)$□ $F[\lambda]$□□□□□□□□□□□□:

$$m(\lambda)=(\lambda-\lambda_1)^{l_1}(\lambda-\lambda_2)^{l_2}\cdots(\lambda-\lambda_s)^{l_s},\tag{1}$$

□ V□□□□□□,□□ A□□□□□□□ A□ Jordan□□□,□□□□□□ A□□□□□□;□□□□□ λ_j □ Jordan□□□□ N_j□

$$N_j=\dim V-\operatorname{rank}(A-\lambda_j I),\quad j=1,2,\cdots,s,\tag{2}$$

□□□□ Jordan□□□□□□□ l_j,t□ Jordan□ $J_t(\lambda_j)$□□□ $N_j(t)$□

$$N_j(t)=\operatorname{rank}(A-\lambda_j I)^{t+1}+\operatorname{rank}(A-\lambda_j I)^{t-1}-2\operatorname{rank}(A-\lambda_j I)^{t};\tag{3}$$

□□ Jordan□□□ A□□ A□ Jordan□□□,□□ Jordan□□□□□□□,A

□ Jordan□□□□□□□.

□□ □(1)□□□, $\lambda_1, \lambda_2, \ldots, \lambda_s$□ A□□□□□□□□□.□(1)□□□

$$V = \mathrm{Ker}\, m(A) = \mathrm{Ker}(A - \lambda_1 I)^{l_1} \oplus \cdots \oplus \mathrm{Ker}(A - \lambda_s I)^{l_s}. \tag{4}$$

□ $W_j = \mathrm{Ker}(A - \lambda_j I)^{l_j}$, $j = 1, 2, \ldots, s$.□ W_j□ A□□□□□□, $j = 1, 2, \ldots, s$.
□(4)□□

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_s. \tag{5}$$

□ $B_j = A|W_j - \lambda_j I$,□§7□□□□□ B_j□ W_j□□□□□□,□□□□□□ l_j.□□□§8□□□3□,□ W_j□□□□□□,□□ B_j□□□□□□□ B_j□ Jordan□□□.□□ $A|W_j$□ W_j□□□□□□□□ $A_j = B_j + \lambda_j I$□□□□□□λ_j□ Jordan□□□.□ W_j $(j = 1, 2, \ldots, s)$□□□□□□□□□□ V□□□□,A□□□□□□□ A□

$$A = \mathrm{diag}\{A_1, A_2, \ldots, A_s\}. \tag{6}$$

□□ A□ Jordan□□□,□□□□□ A□□□□□□.

□□ B_j□□□□□□ l_j,□□ B_j□□□ Jordan□□□□□□□ l_j(□§8□□□3),□□ A_j□□□ Jordan□□□□□□□ l_j.□ 1□t□l_j□,A_j□t□ Jordan□□□□ $N_j(t)$□□ B_j□t□ Jordan□□□□,□□

$$\begin{aligned}
N_j(t) &= \mathrm{rank}\, B_j^{t+1} + \mathrm{rank}\, B_j^{t-1} - 2\mathrm{rank}\, B_j^{t} \\
&= \mathrm{rank}(A|W_j - \lambda_j I)^{t+1} + \mathrm{rank}(A|W_j - \lambda_j I)^{t-1} \\
&\quad - 2\mathrm{rank}(A|W_j - \lambda_j I)^{t} \\
&= \dim W_j - \dim \mathrm{Ker}(A|W_j - \lambda_j I)^{t+1} \\
&\quad + \dim W_j - \dim \mathrm{Ker}(A|W_j - \lambda_j I)^{t-1} \\
&\quad - 2 \dim W_j - \dim \mathrm{Ker}(A|W_j - \lambda_j I)^{t} \\
&= 2\dim \mathrm{Ker}(A|W_j - \lambda_j I)^{t} - \dim \mathrm{Ker}(A|W_j - \lambda_j I)^{t+1} \\
&\quad - \dim \mathrm{Ker}(A|W_j - \lambda_j I)^{t-1}.
\end{aligned} \tag{7}$$

□ i□l_j□,□

$$\begin{aligned}
&\alpha \in \mathrm{Ker}(A - \lambda_j I)^{i} \\
\Leftrightarrow\ &(A - \lambda_j I)^{i}\alpha = 0 \\
\Leftrightarrow\ &\alpha \in \mathrm{Ker}(A - \lambda_j I)^{l_j} = W_j\text{□ } (A - \lambda_j I)^{i}\alpha = 0 \\
\Leftrightarrow\ &\alpha \in W_j\text{□ } (A|W_j - \lambda_j I)^{i}\alpha = 0 \\
\Leftrightarrow\ &\alpha \in \mathrm{Ker}(A|W_j - \lambda_j I)^{i},
\end{aligned} \tag{8}$$

□□

$$\mathrm{Ker}(A - \lambda_j I)^{i} = \mathrm{Ker}(A|W_j - \lambda_j I)^{i}. \tag{9}$$

□ $i = l_j + 1$□,(9)□□□□□(□□□□□□□).□□□(7)□□(9)□□

$$N_j(t) = 2\dim \mathrm{Ker}(A - \lambda_j I)^{t} - \dim \mathrm{Ker}(A - \lambda_j I)^{t+1}$$

$$- \dim \operatorname{Ker}(A-\lambda_j I)^{t-1}$$
$$= \operatorname{rank}(A-\lambda_j I)^{t+1} + \operatorname{rank}(A-\lambda_j I)^{t-1}$$
$$- 2\operatorname{rank}(A-\lambda_j I)^{t}. \qquad (10)$$

A_j Jordan N_j B_j Jordan , B_j $(W_j)_0$.

$$\alpha \in (W_j)_0 \Leftrightarrow B_j\alpha = 0\alpha = 0 \Leftrightarrow \alpha \in \operatorname{Ker} B_j,$$

$(W_j)_0 = \operatorname{Ker} B_j$.

$$N_j = \dim (W_j)_0 = \dim \operatorname{Ker} B_j = \dim \operatorname{Ker}(A|W_j - \lambda_j I)$$
$$= \dim \operatorname{Ker}(A-\lambda_j I) = \dim V - \operatorname{rank}(A-\lambda_j I). \qquad (11)$$

A Jordan A, A , λ_j Jordan N_j V $\operatorname{rank}(A-\lambda_j I)$, λ_j t Jordan $N_j(t)$ $(A-\lambda_j I)^{t+1}$, $(A-\lambda_j I)^{t-1}$, $(A-\lambda_j I)^{t}$, Jordan , A Jordan .

X

: $i = l_j + 1$, (9) : (1)

$$m(\lambda)(\lambda-\lambda_j) = (\lambda-\lambda_1)^{l_1}\cdots(\lambda-\lambda_j)^{l_j+1}\cdots(\lambda-\lambda_s)^{l_s}. \qquad (12)$$

V

$$V = \operatorname{Ker}(A-\lambda_1 I)^{l_1} \oplus \cdots \oplus \operatorname{Ker}(A-\lambda_j I)^{l_j+1} \oplus \cdots \oplus \operatorname{Ker}(A-\lambda_s I)^{l_s}. \qquad (13)$$

$\beta \in \operatorname{Ker}(A-\lambda_j I)^{l_j+1}$, (4) , β

$$\beta = \beta_1 + \cdots + \beta_j + \cdots + \beta_s, \qquad (14)$$

$\beta_u \in \operatorname{Ker}(A-\lambda_i I)^{l_u}$, $u = 1, 2, \ldots, s$. $\beta_j \in \operatorname{Ker}(A-\lambda_j I)^{l_j}$, $\beta_j \in \operatorname{Ker}(A-\lambda_j I)^{l_j+1}$. $\beta_j - \beta \in \operatorname{Ker}(A-\lambda_j I)^{l_j+1}$. (14)

$$0 = \beta_1 + \cdots + (\beta_j - \beta) + \cdots + \beta_s. \qquad (15)$$

(15) 0 V (13) . , $\beta_j - \beta = 0$. $\beta = \beta_j \in \operatorname{Ker}(A-\lambda_j I)^{l_j}$. , $\operatorname{Ker}(A-\lambda_j I)^{l_j+1} \subseteq \operatorname{Ker}(A-\lambda_j I)^{l_j}$.

.

$$\operatorname{Ker}(A-\lambda_j I)^{l_j+1} = \operatorname{Ker}(A-\lambda_j I)^{l_j}. \qquad (16)$$

$\eta \in \operatorname{Ker}(A|W_j - \lambda_j I)^{l_j+1}$, $\eta \in W_j$.

$W_j = \operatorname{Ker}(A-\lambda_j I)^{l_j} = \operatorname{Ker}(A|W_j - \lambda_j I)^{l_j}$,

$\eta \in \operatorname{Ker}(A|W_j - \lambda_j I)^{l_j}$.

$$\operatorname{Ker}(A|W_j - \lambda_j I)^{l_j+1} = \operatorname{Ker}(A|W_j - \lambda_j I)^{l_j}. \qquad (17)$$

(16) (17) (9)

$$\operatorname{Ker}(A-\lambda_j I)^{l_j+1} = \operatorname{Ker}(A|W_j - \lambda_j I)^{l_j+1}. \qquad (18)$$

□□□ 1□□□□□□□□:

□□**2** □ A□□ F□□ n□□□,□□ A□□□□□□ $m(\lambda)$□ $F[\lambda]$ □□□□□□□□□□□□:

$$m(\lambda)=(\lambda-\lambda_1)^{l_1}(\lambda-\lambda_2)^{l_2}\cdots(\lambda-\lambda_s)^{l_s}, \tag{19}$$

□ A□□□□□ Jordan□□□,□□□□□□□□□ A□□□□□□;□□□ □□ λ_j □ Jordan□□□ N_j □

$$N_j=n-\mathrm{rank}(A-\lambda_j I),\quad j=1,2,\dots,s, \tag{20}$$

□□□□□ Jordan□□□□□□□ l_j, t □ Jordan□□□□ $N_j(t)$□

$$N_j(t)=\mathrm{rank}(A-\lambda_j I)^{t+1}+\mathrm{rank}(A-\lambda_j I)^{t-1}-2\mathrm{rank}(A-\lambda_j I)^{t}; \tag{21}$$

□□ Jordan□□□□□ A□ Jordan□□□,□□ Jordan□□□□□□□, A□ Jordan□□□□□□□. x

□□□□□□□□□□□□ 0□□□□□□□□□□□□□□□□□□ □,□□□□□□□□□□□□□□□□□□□□□□□ Jordan□□□,□□ □□□□□□□□□□□ Jordan□□□.

□ **1** □□□□□□□□□□□□□□□:

$$A=\begin{pmatrix}2 & 3 & 2\\ 1 & 8 & 2\\ -2 & -14 & -3\end{pmatrix}.$$

□ A□□□□□□ $f(\lambda)$□

$$|\lambda I-A|=\begin{vmatrix}\lambda-2 & -3 & -2\\ -1 & \lambda-8 & -2\\ 2 & 14 & \lambda+3\end{vmatrix}=(\lambda-1)(\lambda-3)^2,$$

□□ A□□□□□□□ 1, 3(□□).

□□□□□ $\lambda_1=1$,□□ $f(\lambda)$□ 1□□,□□□□ A□□□□□□□□□ □□□□□□ 1□.

□□□□□ $\lambda_2=3$,□□ $\mathrm{rank}(A-3I)$:

$$A-3I=\begin{pmatrix}-1 & 3 & 2\\ 1 & 5 & 2\\ -2 & -14 & -6\end{pmatrix}\to\begin{pmatrix}-1 & 3 & 2\\ 0 & 8 & 4\\ 0 & 0 & 0\end{pmatrix}.$$

□□ $\mathrm{rank}(A-3I)=2$.□□□□□□□ 3□□□□□□□□

$$N(\lambda_2)=3-2=1.$$

□□□□, A□□□□□□□ $\begin{pmatrix}1 & 0 & 0\\ 0 & 3 & 1\\ 0 & 0 & 3\end{pmatrix}$.

□□□ F □□ n □□□ A □□□ Jordan □□□,□□□ Jordan □□□□□□□,A □ Jordan □□□□□□□.□□□□□□□□□□□□:

□□ 1 □ F □□ n □□□ A □□□ Jordan □□□ J,□□ J □□□ Jordan □□□□□□□□□□□□□□□ A □□□□□,□□□ A □□□□□.

□□,□

$$A = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix},$$

□ A □□□□□:$(\lambda - 3)^2, \lambda - 5, \lambda - 5$.

□□ 3 □ F □□□ Jordan □□□□□□□□□□□□□□□□□.

□□ □□□□□□□.

□□□.□□□ Jordan □□□□□□□□($\lambda - a)^t$,□□□□□□□□ a,□□□□ t.□□□□□□

$$\begin{pmatrix} a & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a & 1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & a & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a \end{pmatrix}_{t \times t}.$$

X

□□ 4 □□ n □□□ A□ B □□□□□□□□□□□□□□□.

□□ □□□□ A □ Jordan □□□ J.□□□□□□□□□□□□□□,□□ A□ B □□□□□ J □□ B □ Jordan □□□,□□ A□ B □ Jordan □□□□□ Jordan □□□□□.□□□ 3 □,□□□□□□ A□ B □ Jordan □□□□□ Jordan □□□□□□□□□□□,□ A□ B □□□□□.

X

□□ 4 □□,□□□□□□□□ n □□□□□□□□ $M_n(C)$ □,□□□□□□□□□□□□□□□□.

□□□□□ λ-□□ $\lambda I - A$ □□□□□□□□□□□,□□□□□ A □□□□□.[□]

□□ 2 V □□□□□ A □□□ Jordan □□ V □□□□,□□□ A □□□□□□□□ Jordan □□□.

□ □□□□□□□□□□□(□□)□(□□□□□□□ 1996 □ 12 □ 1 □)□□□□□□□□ 1.

□□□□□□□ A □ Jordan □□□ J □□, □□□□ A □□□ Jordan □, □□□□□□□□ Jordan □□□□□□ S □□□□. □□ $J = S^{-1}AS$, □□ S □□□□□

$$AX = XJ \tag{22}$$

□□□□□□□□□□. □□ $\dim V = n$, □ (22) □ n^2 □□□□ x_{ij} $(i, j = 1, \dots, n)$ □□ n^2 □□□□□□□□□□□, □□□□□□□□, □□□ $X = (x_{ij})$, □□□□□□(□□ A □ Jordan □□□□, □□□□□□(22)□□□□□□□□□), □□□□□□□□ S.

□ 2 □□ 1 □□□□□ A □□□ Jordan □.

□ □ $X = (X_1, X_2, X_3)$, □ A □ Jordan □□□□□□□(22)□

$$A(X_1, X_2, X_3) = (X_1, 3X_2, X_2 + 3X_3), \tag{23}$$

□□
$$AX_1 = X_1, \quad AX_2 = 3X_2, \quad AX_3 = X_2 + 3X_3.$$

□□□□, X_1 □ A □□□ 1 □□□□□□□, □□□□

$$(I - A)Y = 0,$$

□□ $X_1 = (2, 0, -1)'$. □□, X_2 □ A □□□ 3 □□□□□□□, □□□□

$$(3I - A)Y = 0,$$

□□ $X_2 = (1, -1, 2)'$. □□□□□□ $(A - 3I)Y = X_2$,

$$\begin{pmatrix} -1 & 3 & 2 & 1 \\ 1 & 5 & 2 & -1 \\ -2 & -14 & -6 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & -\frac{1}{2} & -1 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

□□□□□□□ $Y_0 = (-1, 0, 0)'$. □ $X_3 = (-1, 0, 0)'$, □

$$X = \begin{pmatrix} 2 & 1 & -1 \\ 0 & -1 & 0 \\ -1 & 2 & 0 \end{pmatrix}. \tag{24}$$

□□□□ X □□□□□. □□□□□ V □□□□□ $\alpha_1, \alpha_2, \alpha_3$ □ A □□□ Jordan □ ξ_1, ξ_2, ξ_3 □□□□□. □□

$$(\xi_1, \xi_2, \xi_3) = (\alpha_1, \alpha_2, \alpha_3) \begin{pmatrix} 2 & 1 & -1 \\ 0 & -1 & 0 \\ -1 & 2 & 0 \end{pmatrix},$$

□ A □□□ Jordan □□:

$$\xi_1 = 2\alpha_1 - \alpha_3, \quad \xi_2 = \alpha_1 - \alpha_2 + 2\alpha_3, \quad \xi_3 = -\alpha_1.$$

□□□ Jordan □□□□□□□□□□□□□□□□□□□□□, □□, □

□□ A^m,□□ m□□□□□□□.□□ A□ Jordan□□□ J,□□□□□□□□□□□□□□ X,□□ $X^{-1}AX=J$,□□

$$A^m=(XJX^{-1})^m=XJ^mX^{-1}. \tag{25}$$

□□ Jordan□ $J_t(\lambda_j)$□□□□□□□:

$$J_t(\lambda_j)^r=(\lambda_j I+J_t(0))^r. \tag{26}$$

□□ $\lambda_j I$□ $J_t(0)$□□□,□□□□□□□□□□(26)□□□.□□:□ r□ t□, $J_t(0)^r=0$.□□□,□□ J^m□□□□□ A^m□□□□□.□□□□□ J□ X□□□□,□□□□□□ m,□□□□□□□□(25)□□ A^m.

□ 3 □□□ 1□□ A,□□ A^{10}.

□ □□ 1□□, A□ Jordan□□□□

$$J=\mathrm{diag}\left(1, \begin{pmatrix}3&1\\0&3\end{pmatrix}\right),$$

□□ 2□□,(24)□□□□□□ X□□ $X^{-1}AX=J$.□□ $A^m=XJ^mX^{-1}$.□□□

$$X^{-1}=\begin{pmatrix}0&-2&-1\\0&-1&0\\-1&-5&-2\end{pmatrix}.$$

□ r□2,□

$$\begin{pmatrix}3&1\\0&3\end{pmatrix}^r=\left(3I+\begin{pmatrix}0&1\\0&0\end{pmatrix}\right)^r$$

$$=3^rI+r3^{r-1}I\begin{pmatrix}0&1\\0&0\end{pmatrix}=\begin{pmatrix}3^r&r3^{r-1}\\0&3^r\end{pmatrix}.$$

□□

$$A^{10}=X\mathrm{diag}\left((1)^{10},\begin{pmatrix}3&1\\0&3\end{pmatrix}^{10}\right)X^{-1}$$

$$=\begin{pmatrix}2&1&-1\\0&-1&0\\-1&2&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&3^{10}&10\cdot 3^9\\0&0&3^{10}\end{pmatrix}\begin{pmatrix}0&-2&-1\\0&-1&0\\-1&-5&-2\end{pmatrix}$$

$$=\begin{pmatrix}-7\cdot 3^9&-38\cdot 3^9-4&-14\cdot 3^9-2\\10\cdot 3^9&53\cdot 3^9&20\cdot 3^9\\-20\cdot 3^9&-106\cdot 3^9+2&-40\cdot 3^9+1\end{pmatrix}.$$

□ □ 9 9

1 □□□□□ K□□□□□ Jordan□□□:

(1) $\begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 1 & 2 \end{pmatrix}$; (2) $\begin{pmatrix} 4 & -5 & 2 \\ 5 & -7 & 3 \\ 6 & -9 & 4 \end{pmatrix}$;

(3) $\begin{pmatrix} 1 & -3 & 3 \\ -2 & -6 & 13 \\ -1 & -4 & 8 \end{pmatrix}$; (4) $\begin{pmatrix} 1 & -3 & 4 \\ 4 & -7 & 8 \\ 6 & -7 & 7 \end{pmatrix}$;

(5) $\begin{pmatrix} 3 & -4 & 0 & 2 \\ 4 & -5 & -2 & 4 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 2 & -1 \end{pmatrix}$; (6) $\begin{pmatrix} 3 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 3 & 0 & 5 & -3 \\ 4 & -1 & 3 & -1 \end{pmatrix}$.

2 □□□ 1□□□□□□□□□,□□□□□□□□□.

3 □□: $J_t(a)$□ $J_t(a)$□.

4 □□:□□ n□□□□ A□□□□□ A□□□.

5 □□:□ n□□□□ A□ n□□□□□$\lambda_1,\lambda_2,\ldots,\lambda_n$,□□□□□□□□ m, A^m□ n □□□□□ $\lambda_1^m,\lambda_2^m,\ldots,\lambda_n^m$.

6 □□:□ n□□□□ A□ n□□□□□$\lambda_1,\lambda_2,\ldots,\lambda_n$,□□□□□□□□□□□ $g(x)$,□□□ $g(A)$□ n□□□□□ $g(\lambda_1), g(\lambda_2),\ldots, g(\lambda_n)$.

7. □

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

(1) □□:□ k□3□,□

$$A^k = A^{k-2} + A^2 - I;$$

(2) □ A^{100}.

§10 □□□□□□□□□

□□□§1□□□□□□,□□□□□□□□□□[a, b]□□□□□□□□ $f(x)$□□□□□□□ $\int_a^b f(x)\,dx$;□□□□□□□□□□□□□□□□□□.

□□□□□□ F□□□□□ n□□□ $A=(a_{ij})$□□□ F□□□□□ $\sum_{i=1}^n a_{ii}$,□□□

$$\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B), \quad \forall A, B \in M_n(F),$$

$$\mathrm{tr}(kA) = k\,\mathrm{tr}(A), \quad \forall A \in M_n(F),\ k \in F.$$

□□ 1 □ V□□ F□□□□□□□□, V□ F□□□□□ f□□□□

$$f(\alpha+\beta)=f(\alpha)+f(\beta),\qquad \forall\,\alpha,\beta\in V,$$
$$f(k\alpha)=kf(\alpha),\qquad \forall\,\alpha\in V,\ k\in F,$$

□□ f □ V □□□□□□□□.

□□□ F □□□□□□□□□□□□□, □□ V □□□□□□□□□□ F □□□□□ V □□□□□ F □□□□□. □□□□ §1 □ §3 □□□□□□□□□□□□□□□□□□.

□□□□□□□□□□□□□□. □□□□□□□□.

□ 1 □□□ F □ n □□□ $a_1, a_2, \dots, a_n$, □

$$f:\quad F^n \qquad\qquad F$$
$$(x_1, x_2, \dots, x_n) \longmapsto a_1x_1+a_2x_2+\dots+a_nx_n,$$

□□□□□□ f □ F^n □□□□□□□□.

□ 2 □

$$f:\quad Z_2^2 \qquad\qquad Z_2$$
$$(x_1, x_2) \longmapsto x_1^2+x_2^2,$$

□□ f □□□ Z_2^2 □□□□□□□□.

□ □□ $(x_1,x_2),(y_1,y_2)$ □ Z_2^2, □

$$f\ (x_1,x_2)+(y_1,y_2)\ =f(x_1+y_1,\ x_2+y_2)$$
$$=(x_1+y_1)^2+(x_2+y_2)^2=x_1^2+y_1^2+x_2^2+y_2^2$$
$$=f(x_1,x_2)+f(y_1,y_2).$$
$$f\ 1\cdot(x_1,x_2)\ =f(x_1,x_2)=1\ f(x_1,x_2),$$
$$f\ 0\ (x_1,x_2)\ =f(0,0)=0=0\ f(x_1,x_2).$$

□□ f □ Z_2^2 □□□□□□□□.

□ V □□ F □ n □□□□□, f □ V □□□□□□□□. □□ f □□□□□ V □ F □□□□□, □□ f □□□□□ V □□□□ $\alpha_1,\alpha_2,\dots,\alpha_n$ □□□□□□□. □□□□□ $f(\alpha_1), f(\alpha_2), \dots, f(\alpha_n)$, □□□□□ V □□□□□ $\alpha=\sum_{i=1}^{n}x_i\alpha_i$ □ f □□□:

$$f(\alpha)=\sum_{i=1}^{n}x_if(\alpha_i).\tag{1}$$

(1) □□□□□□□ f □□ $\alpha_1,\alpha_2,\dots,\alpha_n$ □□□□□.

□□, □ V □□□□□ $\alpha_1,\alpha_2,\dots,\alpha_n$, □ F □□□□□ n □□□ $a_1, a_2, \dots, a_n$. □□ V □□□□□ $\alpha=\sum_{i=1}^{n}x_i\alpha_i$, □□

$$f(\alpha) = \sum_{i=1}^{n} x_i a_i, \tag{2}$$

□§1□□□1□,f□V□□□□□□□□,□□□:

$$f(\alpha_i) = a_i,\ i = 1, 2, \ldots, n. \tag{3}$$

□□□□(3)□□V□□□□□□□□□□.

□V□□F□□□□□□□□,□□V□□□□□□□□□□□□V□F□□□□□,□□□§1□□□□□□□□□□,V□□□□□□□□□□□Hom(V, F)□□F□□□□□□□□,□□□V□□□□□□□□.□□V□□□□n,□

$$\dim Hom(V, F) = \dim V \cdot \dim F = n \cdot 1 = n \tag{4}$$

□□

$$Hom(V, F) \cong V. \tag{5}$$

□V□□□□□$\alpha_1, \alpha_2, \ldots, \alpha_n$,□□□□(2)□□(3)□□□□□□□V□□n□□□□□:□□i□$\{1, 2, \ldots, n\}$,□

$$f_i(\alpha_j) = \begin{cases} 1, & \text{□ } j = i, \\ 0, & \text{□ } j \text{□} i, \end{cases} \tag{6}$$

$$f_i\left(\sum_{j=1}^{n} x_j \alpha_j\right) = \sum_{j=1}^{n} x_j f_i(\alpha_j) = x_i, \tag{7}$$

□f_i□V□□□□□□□□.□□□□□□V□□n□□□□□$f_1, f_2, \ldots, f_n$.□

$$k_1 f_1 + k_2 f_2 + \ldots + k_n f_n = 0,$$

□
$$(k_1 f_1 + k_2 f_2 + \ldots + k_n f_n)\alpha_j = 0,\ j = 1, 2, \ldots, n$$

□□□□.□□□□□□□□□□□□□□□□

$$k_1 f_1(\alpha_j) + \ldots + k_j f_j(\alpha_j) + \ldots + k_n f_n(\alpha_j) = 0,\ j = 1, 2, \ldots, n.$$

□□□□

$$k_j = 0,\ j = 1, 2, \ldots, n.$$

□□$f_1, f_2, \ldots, f_n$□□□□.□□$\dim Hom(V, F) = n$,□□$f_1, f_2, \ldots, f_n$□Hom(V, F)□□□□.□□□□(6)□□(7)□□□□$f_1, f_2, \ldots, f_n$□□□$\alpha_1, \alpha_2, \ldots, \alpha_n$□□□□,□V□□□□□□□□Hom(V, F)□□V□□□□□,□□□V^*.

□V□□F□□□□n□□□□□,V□□□□□$\alpha_1, \alpha_2, \ldots, \alpha_n$,□□□□□$f_1, f_2, \ldots, f_n$□$V^*$□□□□.□□$\alpha = \sum_{j=1}^{n} x_j \alpha_j$□V,□(7)□□

$$f_i(\alpha) = f_i\left(\sum_{j=1}^{n} x_j \alpha_j\right) = x_i. \tag{8}$$

(8) , α $\alpha_1, \alpha_2, \ldots, \alpha_n$ i i

f_i α $f_i(\alpha)$. $f = \sum_{i=1}^{n} k_i f_i$ V^* ,

$$f(\alpha_j) = \sum_{i=1}^{n} k_i f_i(\alpha_j) = k_j. \tag{9}$$

(9) , f α_j $f(\alpha_j)$ f $f_1, f_2, \ldots, f_n$

j .

1 V F n , V : $\alpha_1, \alpha_2, \ldots, \alpha_n$ $\beta_1, \beta_2, \ldots, \beta_n$, : $f_1, f_2, \ldots, f_n$ $g_1, g_2, \ldots, g_n$. V $\alpha_1, \alpha_2, \ldots, \alpha_n$ $\beta_1, \beta_2, \ldots, \beta_n$ A, V^* $f_1, f_2, \ldots, f_n$ $g_1, g_2, \ldots, g_n$ B

$$B = (A^{-1})' . \tag{10}$$

$(\beta_1, \beta_2, \ldots, \beta_n) = (\alpha_1, \alpha_2, \ldots, \alpha_n) A$

$$(\alpha_1, \alpha_2, \ldots, \alpha_n) = (\beta_1, \beta_2, \ldots, \beta_n) A^{-1}. \tag{11}$$

(11) , α_j $\beta_1, \beta_2, \ldots, \beta_n$ A^{-1} j , i $A^{-1}(i; j)$. $\beta_1, \beta_2, \ldots, \beta_n$ $g_1, g_2, \ldots, g_n$, (8) , α_j $\beta_1, \beta_2, \ldots, \beta_n$ i $g_i(\alpha_j)$. $A^{-1}(i; j) = g_i(\alpha_j)$.

$\alpha_1, \alpha_2, \ldots, \alpha_n$ $f_1, f_2, \ldots, f_n$, (9) , $g_i(\alpha_j)$ g_i $f_1, f_2, \ldots, f_n$ j .

$$(g_1, g_2, \ldots, g_n) = (f_1, f_2, \ldots, f_n) B, \tag{12}$$

(12) , g_i $f_1, f_2, \ldots, f_n$ j $B(j; i)$. $g_i(\alpha_j) = B(j; i)$.

$$A^{-1}(i; j) = g_i(\alpha_j) = B(j; i) = B'(i; j),$$

$i, j = 1, 2, \ldots, n$. $A^{-1} = B'$.

$$B = (A^{-1})' .$$

X

V F n , $V \cong V^*$. V $\alpha_1, \alpha_2, \ldots, \alpha_n$, $f_1, f_2, \ldots, f_n$ V^* .

$$\sigma: \quad V \quad\quad V^*$$

$$\alpha = \sum_{i=1}^{n} x_i \alpha_i \longmapsto \sum_{i=1}^{n} x_i f_i, \tag{13}$$

8 § 3 1 , σ V V^* . $\sigma(\alpha) = f_\alpha$. $\beta = \sum_{i=1}^{n} y_i \alpha_i$ V, (8)

$$f_\alpha(\beta) = \Bigl(\sum_{i=1}^n x_i f_i\Bigr)(\beta) = \sum_{i=1}^n x_i f_i(\beta) = \sum_{i=1}^n x_i y_i. \tag{14}$$

(14)□□□,α□σ□□□f_α□β□□□□□$f_\alpha(\beta)$□□α□β□□□□□□□□□□□□.

□□□□□□□□F□□□n□□□□□□□,□□□□□F□n□□□□□V□□□□□V^*,□□□□□□□□□□□$(V^*)^*$,□□□V^{**}.□V^{**}□V□□□□□□□.□□□

$$V \cong V^* \cong V^{**}. \tag{15}$$

□$\alpha_1,\alpha_2,\dots,\alpha_n$□V□□□□,□□□□□$f_1,f_2,\dots,f_n$.□(13)□,V□$V^*$□□□□□□□$\sigma_1$,V□□□$\alpha=\sum_{i=1}^n x_i\alpha_i$□$\sigma_1$□□□□□$f_\alpha$.□□,□$V^*$□$V^{**}$□□□□□□□$\sigma_2$,$f_\alpha$□$\sigma_2$□□□□□$\alpha^{**}$.□□$f \in V^*$,□(14)□□□□□□□,$\alpha^{**}(f)$□□$f_\alpha$□$f$□□$f_1,f_2,\dots,f_n$□□□□□□□□□□□□.□(13)□□(9)□□□□□□□

$$f_\alpha = \sum_{i=1}^n x_i f_i, \quad f = \sum_{i=1}^n f(\alpha_i) f_i, \tag{16}$$

□□

$$\alpha^{**}(f) = \sum_{i=1}^n x_i f(\alpha_i) = f\Bigl(\sum_{i=1}^n x_i\alpha_i\Bigr) = f(\alpha). \tag{17}$$

□$\tau=\sigma_2\sigma_1$,□τ□V□V^{**}□□□□□□□,□□

$$\tau(\alpha) = \sigma_2(\sigma_1(\alpha)) = \sigma_2(f_\alpha) = \alpha^{**}, \quad \forall \alpha \in V, \tag{18}$$

□(17)□□□

$$\alpha^{**}(f) = f(\alpha), \quad \forall f \in V^*. \tag{19}$$

□(18)□□(19)□□□,α□V□V^{**}□□□□□τ□□□α^{**}□□□□V□□□□□.□□□□□□□□□□□□□□□□□□□□.

□□V□V^{**}□□□□□□τ,□□□□□α□$\tau(\alpha)=\alpha^{**}$□□,□□□□□V□V^{**}□□.□□□□□V□□V^*□□□□□.□□V□V^*□□□□□□□.□□□□□□□V^*□□V□□□□□□□□.

□□ 9 10

1 □ $V=C[a,b]$, V□□□□□□□□□□□□□□?

(1) $f(x) \longmapsto \int_a^b f^2(x)\,dx$;

(2) $f(x) \longmapsto \int_a^b f(x)g(x)dx$,

□□ $g(x)$□□□□□□□□.

2 □ V□□ F□□□□ 3□□□□□,$\alpha_1,\alpha_2,\alpha_3$ □ V□□□□,f□ V□□□□□□□□,□□

$$f(\alpha_1+2\alpha_3)=4,\quad f(\alpha_2+3\alpha_3)=0,\quad f(4\alpha_1+\alpha_2)=5,$$

□ $f(x_1\alpha_1+x_2\alpha_2+x_3\alpha_3)$.

3 V□$\alpha_1,\alpha_2,\alpha_3$□□ 2□,□□□□□□□□□□ f,□

$$f(3\alpha_1+\alpha_2)=2,\quad f(\alpha_2-\alpha_3)=1,\quad f(2\alpha_1+\alpha_3)=2.$$

*4 □ V□□ F□□□□□□□,char $F=0$.□ $f_1,f_2,\ldots,f_s$□□ V□□□□□□,□□□□□□□□□□.□□:□□ α□ V,□□

$$f_i(\alpha)□0,\quad i=1,2,\ldots,s.$$

5 □ V□□□ K□□□ 3□□□□□,$\alpha_1,\alpha_2,\alpha_3$ □ V□□□□,□□□□□□ f_1, f_2,f_3.□

$$\beta_1=2\alpha_1+\alpha_2+2\alpha_3,\quad \beta_2=\alpha_1+2\alpha_2-2\alpha_3,\quad \beta_3=-2\alpha_1+2\alpha_2+\alpha_3,$$

□□:β_1,β_2,β_3□ V□□□□,□□□□□□□□(□ f_1,f_2,f_3□□).

6 □ $V=\mathbf{R}[x]_3$,□□ $g(x)$□ V,□□

$$f_1(g(x))=\int_0^1 g(x)dx,\quad f_2(g(x))=\int_0^2 g(x)dx,\quad f_3(g(x))=\int_0^{-1} g(x)dx.$$

□□:f_1,f_2,f_3□ V^*□□□□;□□□□ V□□□□ $g_1(x),g_2(x),g_3(x)$,□□ f_1,f_2,f_3□□□□□□.

7. □ A□□ F□ n□□□□□ V□□□□□□□□.

(1) □□:□□ f□ V^*,□ fA□ V^*;

(2) □□ V^*□□□□□□□□□ A^*□

$$f \longmapsto fA,$$

□□ A^*□ V^*□□□□□□;

(3) □ $\alpha_1,\ldots,\alpha_n$□ V□□□□,$f_1,\ldots,f_n$□□□□□□,□□ A□$\alpha_1,\ldots,\alpha_n$□□□□□ A,□□:A^*□ $f_1,\ldots,f_n$□□□□□ A'(□□ A^*□□ A□□□□□).

8 □ V□□ F□□□□□□□,

(1) $f_1,\ldots,f_s$□ V□ s□□□□□,□□□□□□

$$W=\{\alpha□V \mid f_i(\alpha)=0,\ i=1,\ldots,s\}$$

□ V□□□□□□. W□□□□□□ $f_1,\ldots,f_s$□□□□□.

*(2) □□:□ V□□□□□,□ V□□□□□□□□□□□□□□□□□□□□.

[illegible] 10 [illegible]

[illegible], [illegible]. [illegible] 4 [illegible] § 6 [illegible], [illegible] n [illegible] R^n [illegible], [illegible] R^n [illegible]. [illegible], [illegible], [illegible], [illegible].

§ 1 [illegible]

[illegible], R^n [illegible] (α,β) [illegible] R^n [illegible], [illegible]:

1° $(\alpha,\beta)=(\beta,\alpha)$ ([illegible]);

2° $(\alpha+\gamma,\beta)=(\alpha,\beta)+(\gamma,\beta)$ ([illegible]);

3° $(k\alpha,\beta)=k(\alpha,\beta)$ ([illegible]);

4° (α,α) [illegible] 0, [illegible] $\alpha=0$ ([illegible]).

[illegible] 1°, 2°, 3°, [illegible]

$$(k_1\alpha_1+k_2\alpha_2,\beta)=k_1(\alpha_1,\beta)+k_2(\alpha_2,\beta),$$

$$(\alpha,k_1\beta_1+k_2\beta_2)=k_1(\alpha,\beta_1)+k_2(\alpha,\beta_2).$$

[illegible] R^n [illegible], [illegible] F [illegible] V [illegible]:

[illegible] 1 [illegible] V [illegible] F [illegible], $V\times V$ [illegible] F [illegible] f [illegible]: " $\alpha_1,\alpha_2,\beta_1,\beta_2,\alpha,\beta$ [illegible] V, k_1, k_2 [illegible] F, [illegible]

(i) $f(k_1\alpha_1+k_2\alpha_2,\beta)=k_1f(\alpha_1,\beta)+k_2f(\alpha_2,\beta)$;

(ii) $f(\alpha,k_1\beta_1+k_2\beta_2)=k_1f(\alpha,\beta_1)+k_2f(\alpha,\beta_2)$,

[illegible] f [illegible] V [illegible], f [illegible] $f(\alpha,\beta)$.

[illegible] (i) [illegible]: [illegible] β [illegible], [illegible] $\alpha\longmapsto f(\alpha,\beta)$ [illegible] V [illegible], [illegible] β_R;

[illegible] (ii) [illegible]: [illegible] α [illegible], [illegible] $\beta\longmapsto f(\alpha,\beta)$ [illegible] V [illegible],

□□ α_L.

□ 1 □ $V = M_n(F)$, □

$$f(A, B) = \mathrm{tr}(AB), \quad \forall A, B \in V,$$

□□□□ f □ V □□□□□□□□□.

□ 2 □ $V = C[a, b]$, □

$$f(g(x), h(x)) = \int_a^b g(x)h(x)\,dx, \quad \forall g(x), h(x) \in V,$$

□□□□ f □ V □□□□□□□□□.

□ 3 □ F □□□□, □ F^n □, □□

$$\alpha = (a_1, a_2, \ldots, a_n), \quad \beta = (b_1, b_2, \ldots, b_n),$$

□

$$f(\alpha, \beta) = \sum_{i=1}^{n} a_i b_i,$$

□□□□ f □ F^n □□□□□□□□□.

□ V □□ F □ n □□□□□, f □ V □□□□□□□□□. V □□□□□ $\alpha_1, \alpha_2, \ldots, \alpha_n$. □ α, β □□□□□□□□□□ $X = (x_1, x_2, \ldots, x_n)'$, $Y = (y_1, y_2, \ldots, y_n)'$, □

$$f(\alpha, \beta) = f\left(\sum_{i=1}^{n} x_i\alpha_i, \sum_{j=1}^{n} y_j\alpha_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i y_j f(\alpha_i, \alpha_j). \tag{1}$$

□

$$A = \begin{pmatrix} f(\alpha_1, \alpha_1) & f(\alpha_1, \alpha_2) & \cdots & f(\alpha_1, \alpha_n) \\ f(\alpha_2, \alpha_1) & f(\alpha_2, \alpha_2) & \cdots & f(\alpha_2, \alpha_n) \\ \cdots & \cdots & & \cdots \\ f(\alpha_n, \alpha_1) & f(\alpha_n, \alpha_2) & \cdots & f(\alpha_n, \alpha_n) \end{pmatrix}, \tag{2}$$

□ A □□□□□□ f □□ $\alpha_1, \alpha_2, \ldots, \alpha_n$ □□□□□, □□□ f □□ $\alpha_1, \alpha_2, \ldots, \alpha_n$ □□□□□.

□(1)□□

$$f(\alpha, \beta) = X'AY. \tag{3}$$

(3)□□(1)□□□□□□□□ f □□ $\alpha_1, \alpha_2, \ldots, \alpha_n$ □□□□□.

□□□□, □□□□□ $X, Y \in F^n$, □□ $X'AY = X'BY$, □ $A = B$(□□ $\varepsilon_i' A \varepsilon_j = a_{ij}$, □□□□).

□□ 1 □ f □□ F □ n □□□□□ V □□□□□□□□□, V □□□□□: $\alpha_1, \alpha_2, \ldots, \alpha_n$ □ $\beta_1, \beta_2, \ldots, \beta_n$. □

$$(\beta_1, \beta_2, \ldots, \beta_n) = (\alpha_1, \alpha_2, \ldots, \alpha_n)P, \tag{4}$$

f □□□□□□□□□□□□□ A, B, □

$$B = P'AP. \qquad (5)$$

[illegible] $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) X = (\beta_1, \beta_2, \ldots, \beta_n) X_0$, $\beta = (\alpha_1, \alpha_2, \ldots, \alpha_n) Y = (\beta_1, \beta_2, \ldots, \beta_n) Y_0$, [illegible]

$$X = PX_0, \quad Y = PY_0,$$

[illegible] $f(\alpha, \beta) = X'AY = (PX_0)'A(PY_0) = X_0'(P'AP)Y_0$.

[illegible] $f(\alpha, \beta) = X_0' BY_0$, [illegible], $X_0' BY_0 = X_0'(P'AP)Y_0$, [illegible] X_0, Y_0 [illegible] K^n [illegible], [illegible] $B = P'AP$. X

[illegible] 1 [illegible], V [illegible] f [illegible]. [illegible], [illegible] f [illegible] V [illegible] f [illegible], [illegible] $\text{rank}_m f$.

[illegible] **2** [illegible] f [illegible] F [illegible] V [illegible], V [illegible]

$$\{\alpha \in V \mid f(\alpha, \beta) = 0, \ \forall \beta \in V\} \qquad (6)$$

[illegible] f [illegible] V [illegible], [illegible] $\text{rad}_L V$; V [illegible]

$$\{\beta \in V \mid f(\alpha, \beta) = 0, \ \forall \alpha \in V\} \qquad (7)$$

[illegible] f [illegible] V [illegible], [illegible] $\text{rad}_R V$.

[illegible], f [illegible] V [illegible].

[illegible] **4** [illegible] R^3 [illegible], [illegible] $\alpha = (x_1, x_2, x_3)'$, $\beta = (y_1, y_2, y_3)'$. [illegible]

$$f(\alpha, \beta) = x_1 y_1 - x_3 y_3 = (x_1, x_2, x_3) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

[illegible], f [illegible] R^3 [illegible]. [illegible] $\alpha_1 = (0, 1, 0)$, [illegible] R^3 [illegible] $\beta = (y_1, y_2, y_3)'$, [illegible]

$$f(\alpha_1, \beta) = 0 \cdot y_1 - 0 \cdot y_3 = 0.$$

[illegible] α_1 [illegible] f [illegible].

[illegible] **3** [illegible] V [illegible] f [illegible], [illegible] f [illegible].

[illegible] **2** [illegible] f [illegible] F [illegible] n [illegible] V [illegible], f [illegible] $\alpha_1, \alpha_2, \ldots, \alpha_n$ [illegible] A, [illegible] f [illegible] A [illegible].

[illegible] f [illegible] 0 [illegible] A [illegible]. [illegible]

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) X, \quad \beta = (\alpha_1, \alpha_2, \ldots, \alpha_n) Y,$$

[illegible] $f(\alpha, \beta) = X'AY$, [illegible]

$$f \text{ [illegible] } \text{rad}_L(V) = 0$$

$\Leftrightarrow$ □ $f(\alpha,\beta)=0,\ \forall\beta\in V$ □□□□ $\alpha=0$

$\Leftrightarrow$ □ $X'AY=0,\ \forall Y\in F^n$ □□□□ $X=0$

$\Leftrightarrow$ □ $X'A\varepsilon_i=0,\ i=1,2,\dots,n$ □□□□ $X=0$

$\Leftrightarrow$ □ $X'A(\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n)=0$ □□□□ $X=0$

$\Leftrightarrow$ □ $X'AI=0$ □□□□ $X=0$

$\Leftrightarrow$ □ $A'X=0$ □□□□ $X=0$

$\Leftrightarrow$ □□□□□□□ $A'X=0$ □□□□

$\Leftrightarrow$ $\mathrm{rank}(A')=n$

$\Leftrightarrow$ $\mathrm{rank}(A)=n$.

□□□□，f □□□□ 0 □□□□ A □□．□□ f □□□□□□□□□□□□ □□□ A □□．

□□□ 2 □□□□□□□□，f □□□□□ 0 □□□□ f □□□□□ 0.

□□ **4** □ f □□ F □□□□□ V □□□□□□□□□□，□□

$$f(\alpha,\beta)=f(\beta,\alpha),\quad \forall\alpha,\beta\in V, \tag{8}$$

□□ f □□□□□；□□

$$f(\alpha,\beta)=-f(\beta,\alpha),\quad \forall\alpha,\beta\in V, \tag{9}$$

□□ f □□□□□□（□□□□□）.

□□□□□□□□ R^n □□□□□ (α,β) □□□□□□□□．□ 1□□ 2□□ 3 □□□□□□□□□□□□□.

□ **5** □ R^2 □，□□

$$\alpha=(x_1,x_2),\ \beta=(y_1,y_2),$$

□

$$f(\alpha,\beta)=x_1y_2-x_2y_1.$$

□□□□ f □□□□□□□□□.

□□□□□□ f □ V □□ $\alpha_1,\alpha_2,\dots,\alpha_n$ □□□□□□□ A，□□ f □□□ □，□

$$f(\alpha_i,\alpha_j)=f(\alpha_j,\alpha_i),\quad i,j=1,2,\dots,n,$$

□□ A □□□□□．□□，□□ A □□□□□，□□□□ $\alpha=(\alpha_1,\dots,\alpha_n)X$，$\beta=(\alpha_1,\dots,\alpha_n)Y$，□

$$f(\alpha,\beta)=X'AY=(X'AY)'=Y'A'(X')'$$

$$=Y'AX=f(\beta,\alpha),$$

□□ f □□□□．□□□□□□□：

□□□□□ f □□□□ $\Leftrightarrow$ f □□□□□□□□□□□.

□□□□：

□□□□□ f □□□□□ � f □□□□□□□□□□□,

□□ 3 □ f □□□ K □ n □□□□□ V □□□□□□□□□□□,□ V □□□□□□,□□ f □□□□□□□□□□□□□□.

□□ □□ V □□□□ $\alpha_1,\alpha_2,\ldots,\alpha_n$,□ f □□□□□□□□□□□□ A,□ A □□□□□.□□□□□ K □□□□□□□□□□□□□□□□□□,□□ □□ K □□□□□ C,□□ $C'AC=D$,□□ D □□□□□.□

$$(\eta_1,\eta_2,\ldots,\eta_n)=(\alpha_1,\alpha_2,\ldots,\alpha_n)C,$$

□□ C □□□□,□□ $\eta_1,\eta_2,\ldots,\eta_n$ □□ V □□□□.f □□ $\eta_1,\eta_2,\ldots,\eta_n$ □□□□□□

$$C'AC=D.$$ x

□□ 3 □□□□□□□□□□□ 2 □□□□□□□□□□□□□□,□□□□□□□ 4.

□□ 4 □ f □□□□□ 2 □□ F □□ n □□□□□ V □□□□□□□□□,□ V □□□□□□,□ f □□□□□□□□□□□□□□.

*□□ □□□ n □□□□.

$n=1$ □,f □□□□□□□□□□ 1 □□□,□□□□□□.

□□ F □ $n-1$ □□□□□□□□□□□□□□□□□□,□□□ n □□□□□ V □□□□□□□□□ f.

□□□□□ α,β □ V,□□ $f(\alpha,\beta)=0$,□□□□□□□.□□□ $f(\alpha,\beta)$ □□□□.□□□□□□□□□□ α_1 □ 0,□□

$$f(\alpha_1,\alpha_1)\text{□}0.$$

□□,□□□□ α □ V,□□ $f(\alpha,\alpha)=0$,□□□□ α,β □ V,□

$$\begin{aligned}0&=f(\alpha+\beta,\alpha+\beta)=f(\alpha,\alpha+\beta)+f(\beta,\alpha+\beta)\\&=f(\alpha,\alpha)+f(\alpha,\beta)+f(\beta,\alpha)+f(\beta,\beta)\\&=2f(\alpha,\beta).\end{aligned}$$

□□ char F □ 2,□□□□□□ $f(\alpha,\beta)=0$,□□.□ $f(\alpha_1,\alpha_1)=d_1$,□ α_1 □□□ V □□□□ $\alpha_1,\alpha_2,\ldots,\alpha_n$.□□□□ Schimidt □□□□□,□

$$\tilde{\alpha}_i=\alpha_i-\frac{f(\alpha_1,\alpha_i)}{f(\alpha_1,\alpha_1)}\alpha_1,\quad i=2,\ldots,n, \tag{10}$$

□□□□□

$$f(\alpha_1,\tilde{\alpha}_i)=0,\quad i=2,\ldots,n. \tag{11}$$

□(10)□□□□ $\alpha_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n$ □□ V □□□□.□

$$W=\text{□}\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n\text{□},$$

□ $V=$□α_1□� W,□ f □□ W □□□□□□□,□□□□□□.□□□□□, W

□□□□□□$\eta_2,\ldots,\eta_n$,□□ f□□□□□□□□□□□□□□□ D_1,□

$$f(\eta_i,\eta_j)=0,\quad i□j,\quad i,j=2,\ldots,n. \tag{12}$$

□(11)□□□,□□□ β□ W,□ $f(\alpha_1,\beta)=0$.□□ $f(\alpha_1,\eta_i)=0$, $i=2,\ldots,n$.□□ f□ V□□□□$\alpha_1,\eta_2,\ldots,\eta_n$□□□□□□□□□□□□ $D=\mathrm{diag}\{d_1,D_1\}$.

□□□□□,□□□□□□ n,□□□□.

□□□4□□,□□ V□□□□□□□□□□□□□□,□□□ V□□□□$\eta_1,\ldots,\eta_n$,□□

$$f(\eta_i,\eta_i)□0,\quad i=1,\ldots,n,$$

$$f(\eta_i,\eta_j)=0,\quad i□j,\quad i,j=1,\ldots,n.$$

□□4□□□□□□□□□□□□□□□□□□□□□□□□□□□.□□□□□□ V□□□□$\eta_1,\eta_2,\ldots,\eta_n$,□□ f□□□□□□□□□□□□□□□ $\mathrm{diag}\{d_1,\ldots,d_n\}$,□□□□□

$$\alpha=x_1\eta_1+\ldots+x_n\eta_n,\quad \beta=y_1\eta_1+\ldots+y_n\eta_n,$$

□

$$f(\alpha,\beta)=d_1x_1y_1+d_2x_2y_2+\ldots+d_nx_ny_n. \tag{13}$$

□□□□□□□□□□□□□□□□□.□ f□□ F□ n□□□□□ V□□□□□□□□□.

*□□ char $F=2$,□□□□ α,β□ V,□ $2f(\alpha,\beta)=0$.□□ $f(\alpha,\beta)+f(\alpha,\beta)=0$.□□ $f(\alpha,\beta)=-f(\alpha,\beta)$.□□ f□□□□□,□□

$$f(\alpha,\beta)=-f(\beta,\alpha)=f(\beta,\alpha).$$

□□□ f□□□□□.□□,□□□□2□□□□□□□□□,□□□□□□□□□□□□□□□□□□□.

□□□ char F□2.□□□□□ α□ V,□ $f(\alpha,\alpha)=-f(\alpha,\alpha)$.□□ $2f(\alpha,\alpha)=0$.□□□□ $f(\alpha,\alpha)=0$.

□□**5** □ f□□□□□2□□ F□□ n□□□□□ V□□□□□□□□□□,□□□ V□□□□$\varepsilon_1,\varepsilon_{-1},\ldots,\varepsilon_r,\varepsilon_{-r},\eta_1,\ldots,\eta_s$,□□ f□□□□□□□□□□□□□

$$\mathrm{diag}\left(\begin{pmatrix}0&1\\-1&0\end{pmatrix},\ldots,\begin{pmatrix}0&1\\-1&0\end{pmatrix},0,\ldots,0\right), \tag{14}$$

□

$$\begin{aligned}&f(\varepsilon_i,\varepsilon_{-i})=1,\quad && i=1,\ldots,r,\\ &f(\varepsilon_i,\varepsilon_j)=0, && i+j□0,\\ &f(\alpha,\eta_k)=0, && \alpha□V,\quad k=1,\ldots,s.\end{aligned} \tag{15}$$

*□□ □□□ n□□□□.

$n=1$□, f□□□□□□0.

□□□□□□ n□,□□□□.□□ n□□□.

□□ $f=0$,□ f□□□□□□ 0.

□□□ f□0.□□ V□□□□□□□□□□□□ ε_1,α_2,□□ $f(\varepsilon_1,\alpha_2)$□0.
□□□:□□□□□□□□□□□□ α,β,□□

$$f(\alpha,\beta)=0.$$

□□□□□□□□□□□ $\alpha, k\alpha$,□

$$f(\alpha, k\alpha)=kf(\alpha,\alpha)=0.$$

□ $f=0$,□□.□ $\varepsilon_{-1}=f(\varepsilon_1,\alpha_2)^{-1}\alpha_2$,□ $f(\varepsilon_1,\varepsilon_{-1})=1$.□□ $\varepsilon_1,\varepsilon_{-1}$□□□□
□.□ $\varepsilon_1,\varepsilon_{-1}$□□□ V□□□□ $\varepsilon_1,\varepsilon_{-1},\beta_3,\dots,\beta_n$.□

$$\beta_i'=\beta_i-f(\beta_i,\varepsilon_{-1})\varepsilon_1+f(\beta_i,\varepsilon_1)\varepsilon_{-1},\quad i=3,\dots,n, \tag{16}$$

□

$$\begin{aligned} f(\varepsilon_1,\beta_i') &= f(\varepsilon_1,\beta_i)-f(\beta_i,\varepsilon_{-1})f(\varepsilon_1,\varepsilon_1) \\ &\quad +f(\beta_i,\varepsilon_1)f(\varepsilon_1,\varepsilon_{-1}) \\ &= f(\varepsilon_1,\beta_i)+f(\beta_i,\varepsilon_1)=0,\ i=3,\dots,n, \end{aligned} \tag{17}$$

$$\begin{aligned} f(\varepsilon_{-1},\beta_i') &= f(\varepsilon_{-1},\beta_i)-f(\beta_i,\varepsilon_{-1})f(\varepsilon_{-1},\varepsilon_1) \\ &\quad +f(\beta_i,\varepsilon_1)f(\varepsilon_{-1},\varepsilon_{-1}) \\ &= f(\varepsilon_{-1},\beta_i)+f(\beta_i,\varepsilon_{-1})=0,\ i=3,\dots,n. \end{aligned} \tag{18}$$

□□□(16)□□□,$\varepsilon_1,\varepsilon_{-1},\beta_3',\dots,\beta_n'$□□ V□□□□,□

$$W=L(\beta_3',\dots,\beta_n'),$$

□

$$V=L(\varepsilon_1,\varepsilon_{-1})\oplus W.$$

□ f□□ W□□□□□□,□□□□□□□.□□□□□,□□ W□□□□

$$\varepsilon_2,\varepsilon_{-2},\dots,\varepsilon_r,\varepsilon_{-r},\eta_1,\dots,\eta_s,$$

□□ f□□□□□□□□□□□

$$B_1=\mathrm{diag}\left(\begin{pmatrix}0&1\\-1&0\end{pmatrix},\dots,\begin{pmatrix}0&1\\-1&0\end{pmatrix},0,\dots,0\right). \tag{19}$$

□(17)□(18)□□□,$\forall\beta$□ W,□ $f(\varepsilon_1,\beta)=0$□ $f(\varepsilon_{-1},\beta)=0$.□□ f□ V
□□□□

$$\varepsilon_1,\varepsilon_{-1},\varepsilon_2,\varepsilon_{-2},\dots,\varepsilon_r,\varepsilon_{-r},\eta_1,\dots,\eta_s$$

□□□□□□□

$$B=\mathrm{diag}\left(\begin{pmatrix}0&1\\-1&0\end{pmatrix},\begin{pmatrix}0&1\\-1&0\end{pmatrix},\dots,\begin{pmatrix}0&1\\-1&0\end{pmatrix},0,\dots,0\right).$$

□□□□□,□□□□□ n,□□□□. ∎

□□ 5□□,□□ V□□□□□□□□□□ f□□□□□,□□□ V□
□□□,□□ f□□□□□□□□□□

$$\mathrm{diag}\left\{\begin{pmatrix}0&1\\-1&0\end{pmatrix},\begin{pmatrix}0&1\\-1&0\end{pmatrix},\cdots,\begin{pmatrix}0&1\\-1&0\end{pmatrix}\right\}. \qquad (20)$$

□□□□, $\dim V$ □□□□□.

□□ 10.1

1 □ K^4 □,□ $\alpha=(x_1,x_2,x_3,x_4)$□, $\beta=(y_1,y_2,y_3,y_4)$□,□

$$f(\alpha,\beta)=x_1y_1+x_2y_2+x_3y_3-x_4y_4,$$

(1) □□: f □ K^4 □□□□□□□□□;

(2) □ f □□□□ $\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4$ □□□□□□;

(3) □□ f □□□□□;

(4) □□ f □□□□;

(5) □□□□□ $\alpha\neq 0$, □□ $f(\alpha,\alpha)=0$.

2 □□: □ V □□□□□□□□□, V □□□□□□ f □□□□□□□□□□, □□□ $\dim V-\mathrm{rank}_m f$.

3 □ f □□ F □ n □□□□□ V □□□□□□□□□.

□□: (1) □□ $L_f:\alpha\longmapsto\alpha_L$ □ V □ V^* □□□□□□□; □□ $R_f:\beta\longmapsto\beta_R$ □ V □ V^* □□□□□□□;

(2) $\mathrm{Ker}\,L_f=\mathrm{rad}_L V$, $\mathrm{Ker}\,R_f=\mathrm{rad}_R V$;

(3) $\mathrm{rank}\,L_f=\mathrm{rank}_m f=\mathrm{rank}\,R_f$;

(4) f □□□□□□□□□□□□ L_f(□ R_f)□□□□□ V □ V^* □□□□□.

4 □□: $M_n(F)$ □□□□□□□ $f(A,B)=\mathrm{tr}(AB)$ □□□□□.

5 □ V □□□□□□ n □□□□□, $n\geq 2$. □ f □ V □□□□□□□□□□□□.

(1) □□: V □□□□□□ ξ, □ $f(\xi,\xi)=0$;

(2) □□ f □□□□□, □□□□□□□□□ ξ,η, □□: $f(\xi,\eta)=1$, $f(\xi,\xi)=f(\eta,\eta)=0$.

6 □ f □□□□ 2 □□ F □□□□ V □□□□□, □□: f □□□□□□□□□□□: □□□ $\alpha\in V$, □□ $f(\alpha,\alpha)=0$.

7. □ V □□ F □□□□□, f □ V □□□□□□□□□□□□□. □ W □ V □□□□□, □

$$W^{\perp}\overset{\mathrm{def}}{=}\{\alpha\in V\mid f(\alpha,\beta)=0,\ \forall\beta\in W\}.$$

□□: $W^{\perp}$ □ V □□□□, □ $W^{\perp}$ □ W □□□□.

8 □ V □□ F □□ n □□□□□, f □ V □□□□□□□□□□□□, W □ V □□□□□. □□:

(1) $\dim W+\dim W^{\perp}=\dim V$;

(2) $(W^{\perp})^{\perp}=W$.

§2 □□□□□□

□□□□□□,R^n □□□□□□(α,β)□ R^n □□□□□□□□□□,□□□□□□,□□,□□□□□□,□□(α,β)□□□□□□□□□□□□□.□□□□□□,□□□□□□□□:

□□ 1 □ V□□□□ R□□□□□□□□,V□□□□□□□□□□□□□ f□□ V□□□□□□(f□□□□□□□," α□ V,□□ $f(\alpha,\alpha)$□0,□□□□□□□ $\alpha=0$).

□□□□□□ f□□□□□□(α,β)□□□□□ $f(\alpha,\beta)$□□□(α,β).

□□ 1 □ V□ R□□□□ n□□□□□,V□□□□□□□ f□□□□□□□□□□□□□,f□ V□□□□□□□□□□□□□□□□.

□□ V□□□□□$\alpha_1,\alpha_2,\ldots,\alpha_n$,□ f□□□□□□□□□□ A;α,β□□□□□□□□□□ X, Y,□ $f(\alpha,\beta)=X$□AY.

f□□□□□□□□□□□□□ A□□□□□.

f□□□□ � " α□ V,□ α□0,□ $f(\alpha,\alpha)>0$

� " X□ R^n,□ X□0,□ X□$AX>0$

� A□□□□□. x

□ 1 R^3□,□ $\alpha=(a_1,a_2,a_3)$,$\beta=(b_1,b_2,b_3)$,□□

$$(\alpha,\beta)\stackrel{\text{def}}{=}a_1b_1+2a_2b_2+3a_3b_3, \tag{1}$$

□□□□,(α,β)□ R^3□□□□□□,□□ R^3□□□□□□□□.

□ 2 □□□□ R□□□□□□ $M_n(R)$□,□□□□□□□□□

$$(A,B)\stackrel{\text{def}}{=}\mathrm{tr}(AB□), \tag{2}$$

□□□□,(A,B)□ $M_n(R)$□□□□□□.

□ 3 □ $C[a,b]$□,□□□□□□□□□

$$(f,g)\stackrel{\text{def}}{=}\int_a^b f(x)g(x)\,dx, \tag{3}$$

□□□□,(f,g)□ $C[a,b]$□□□□□□.

□□ 2 □□□ R□□□□□□ V□□□□□□□□□,□□ V□□□□□□□□.□□□□□□□□□ V□□□□□□□□,□□□□□□□ V□□□□□□□□□□□ V□□□.

□□□□□□ V□,□□□□□□□□□,□□□□□□□□□□□□□□□□□□□□□.

□□ 3 □□□□ (α,α) □□□□ α □□□, □□ $|\alpha|$ (□□ $\|\alpha\|$).

□□□□□□□□,□□□□□□□ 0,□□□□□□□□□□.□□□

$$|k\alpha| = |k||\alpha|, \quad \forall \alpha \in V,\ k \in \mathbb{R}.$$

□□ $|k\alpha| = (k\alpha, k\alpha) = k^2(\alpha,\alpha) = |k||\alpha|.$ x

□□□ 1 □□□□□□□□□.□□ $\alpha \neq 0$, □ $\frac{1}{|\alpha|}\alpha$ □□□□□□□.□ α □□ $\frac{1}{|\alpha|}\alpha$ □□□ α □□□.

□□ 2 (Cauchy-Buniakowski □□□) □□□□□ V □,□□□□□□ α, β, □

$$|(\alpha,\beta)| \le |\alpha||\beta|, \tag{4}$$

□□□□□□□□ α, β □□□□.

□□ □□ α, β □□□□, □ $\alpha = 0$ □□ $\beta = k\alpha$. □□ $\alpha = 0$, □

$$|(0,\beta)| = |0(0,\beta)| = 0 = |0||\beta|;$$

□□ $\beta = k\alpha$, □

$$|(\alpha,\beta)| = |(\alpha, k\alpha)| = |k(\alpha,\alpha)| = |k||\alpha|^2 = |\alpha||k\alpha| = |\alpha||\beta|.$$

□□ α, β □□□□,□□□□□ t,□ $\beta \neq t\alpha$,□□□ $t\alpha - \beta \neq 0$.□□□□□□□□□, $\forall t \in \mathbb{R}$, □

$$0 < (t\alpha - \beta, t\alpha - \beta) = t^2|\alpha|^2 - 2t(\alpha,\beta) + |\beta|^2, \tag{5}$$

□□ (5) □□□ t □ 2 □□□□□□□□□□ 0,□

$$4(\alpha,\beta)^2 - 4|\alpha|^2|\beta|^2 < 0.$$

□□□□ $$|(\alpha,\beta)| < |\alpha||\beta|.$$ x

□□ 4 □□□□□□,□□□□□□ α, β □□□ $\langle\alpha,\beta\rangle$ □□□

$$\langle\alpha,\beta\rangle \overset{\text{def}}{=} \arccos\frac{(\alpha,\beta)}{|\alpha||\beta|}, \tag{6}$$

□□ $0 \le \langle\alpha,\beta\rangle \le \pi$.

□ (6) □□□

$$\langle\alpha,\beta\rangle = \frac{\pi}{2} \Leftrightarrow (\alpha,\beta) = 0,$$

□□,□

□□ 5 □□ $(\alpha,\beta) = 0$, □□ α □ β □□,□□ $\alpha \perp \beta$.

□□ 3 □□□□□ V □,□□□□□□□□,□□□□□ $\alpha, \beta \in V$, □

$$|\alpha + \beta| \le |\alpha| + |\beta|. \tag{7}$$

□□ $$|\alpha+\beta|^2 = (\alpha+\beta, \alpha+\beta) = |\alpha|^2 + 2(\alpha,\beta) + |\beta|^2$$
$$\le |\alpha|^2 + 2|\alpha||\beta| + |\beta|^2 = (|\alpha| + |\beta|)^2.$$

□□□□ $$|\alpha+\beta| \,\square\, |\alpha|+|\beta|.$$ ×

□□ 4 □□□□□□ V□, □□□□□□, □□□ α □ β □□, □

$$|\alpha+\beta|^2 = |\alpha|^2+|\beta|^2. \tag{8}$$

□□ $|\alpha+\beta|^2 = (\alpha+\beta, \alpha+\beta) = |\alpha|^2+|\beta|^2.$ ×

□□ 6 □□□□□□ V□, □□□□□ α, β □ V, □□

$$d(\alpha, \beta) \overset{\text{def}}{=} |\alpha-\beta|, \tag{9}$$

□ $d(\alpha,\beta)$ □ α □ β □□□.

□□□□, □□□ α,β,γ □ V, □

1° $d(\alpha,\beta) = d(\beta,\alpha)$ (□□□);

2° $d(\alpha,\beta) \,\square\, 0$, □□□□□□□□ $\alpha=\beta$ (□□□);

3° $d(\alpha,\gamma) \,\square\, d(\alpha,\beta)+d(\beta,\gamma)$ (□□□□□□).

□□□□□□□ V□, □□□□□□□□□□□□□□□□□□□□□□□□. □□□□□□□□□□□□□□□□□□□□□□□□□□□.

□□ 5 □□□□□□□ V□, □□□□□□□□□□□□□□□.

□□ □□ 4 □ § 6 □□□ 2 □□□□□. ×

□ n □□□□□□□ V□, n □□□□□□□□□□□□□□□□ V □□□□, □□□□□□; n □□□□□□□□□□□□□□□ V □□□□□□□□ (□□□□□□).

□□ 4 □ § 6 □□□ 4 □□□□□□□□□, □□□□□□□□:

□□ 6 □ $\alpha_1,\alpha_2,\ldots,\alpha_s$ □□□□□□□ V □□□□□□□□□□□□, □

$$\beta_1 = \alpha_1,$$

$$\beta_2 = \alpha_2 - \frac{(\alpha_2,\beta_1)}{(\beta_1,\beta_1)}\beta_1,$$

$$\cdots\cdots\cdots\cdots$$

$$\beta_s = \alpha_s - \sum_{j=1}^{s-1}\frac{(\alpha_s,\beta_j)}{(\beta_j,\beta_j)}\beta_j, \tag{10}$$

□ $\beta_1,\beta_2,\ldots,\beta_s$ □□□□□□, □□ $\beta_1,\beta_2,\ldots,\beta_s$ □ $\alpha_1,\alpha_2,\ldots,\alpha_s$ □□. ×

□□ 6 □□□□□□□□□□ $\alpha_1,\alpha_2,\ldots,\alpha_s$ □□□□□□□□□□□□ $\beta_1,\beta_2,\ldots,\beta_s$ □□□□□□□□ (Schimidt) □□□. □□□□□□ β_j □□□, □□□□□□□ $\alpha_1,\alpha_2,\ldots,\alpha_s$ □□□□□□□□□□ $\eta_1,\eta_2,\ldots,\eta_s$. □□□ n □□□□□□□ V □□□□ $\alpha_1,\alpha_2,\ldots,\alpha_n$ □□□□□□□□□□, □□□□□, □□□□ V □□□□□□□□ $\eta_1,\eta_2,\ldots,\eta_n$.

□□□□□, n □□□□□□ V□, □□□ $\eta_1,\eta_2,\ldots,\eta_n$ □ V □□□□□□□□□□□

$$(\eta_i,\eta_j)=\delta_{ij},\quad i,j=1,2,\dots,n. \tag{11}$$

[illegible]. [illegible] α,β [illegible] V [illegible] $\eta_1,\eta_2,\dots,\eta_n$ [illegible] $X=(x_1,x_2,\dots,x_n)'$, $Y=(y_1,y_2,\dots,y_n)'$, [illegible]

$$(\alpha,\beta)=\left(\sum_{i=1}^{n}x_i\eta_i,\ \sum_{j=1}^{n}y_j\eta_j\right)=\sum_{i=1}^{n}\sum_{j=1}^{n}x_iy_j(\eta_i,\eta_j)$$

$$=\sum_{i=1}^{n}x_iy_i=X'Y. \tag{12}$$

[illegible], [illegible]. [illegible] α [illegible] $\eta_1,\eta_2,\dots,\eta_n$ [illegible] $(x_1,x_2,\dots,x_n)'$, [illegible]

$$\alpha=\sum_{i=1}^{n}x_i\eta_i.$$

[illegible] η_j [illegible], [illegible]

$$(\alpha,\eta_j)=\left(\sum_{i=1}^{n}x_i\eta_i,\ \eta_j\right)=\sum_{i=1}^{n}x_i(\eta_i,\eta_j)=x_j,$$

[illegible]

$$\alpha=\sum_{i=1}^{n}(\alpha,\eta_i)\eta_i. \tag{13}$$

(13) [illegible] α [illegible] (Fourier) [illegible], [illegible] (α,η_i) [illegible] α [illegible] (Fourier) [illegible].

[illegible] 7 [illegible] V [illegible], [illegible].

[illegible] $\eta_1,\eta_2,\dots,\eta_n$ [illegible] $\beta_1,\beta_2,\dots,\beta_n$ [illegible] V [illegible], P [illegible] $\eta_1,\eta_2,\dots,\eta_n$ [illegible] $\beta_1,\beta_2,\dots,\beta_n$ [illegible], [illegible]

$$(\beta_1,\beta_2,\dots,\beta_n)=(\eta_1,\eta_2,\dots,\eta_n)P, \tag{14}$$

[illegible] P [illegible] j [illegible] Y_j [illegible] β_j [illegible] $\eta_1,\eta_2,\dots,\eta_n$ [illegible], $j=1,2,\dots,n$, [illegible]

$$(\beta_i,\beta_j)=Y_i'Y_j,\quad i,j=1,2,\dots,n. \tag{15}$$

[illegible] $\beta_1,\beta_2,\dots,\beta_n$ [illegible], [illegible]

$$(\beta_i,\beta_j)=\delta_{ij},\quad i,j=1,2,\dots,n. \tag{16}$$

[illegible] (15) [illegible] (16) [illegible]

$$Y_i'Y_j=\delta_{ij},\quad i,j=1,2,\dots,n. \tag{17}$$

[illegible] 4 [illegible] § 6 [illegible] 1 [illegible], P [illegible]. x

[illegible] 8 n [illegible] V [illegible], [illegible] $\eta_1,\eta_2,\dots,\eta_n$ [illegible] V [illegible], [illegible] $\beta_1,\beta_2,\dots,\beta_n$ [illegible]

$$(\beta_1,\beta_2,\dots,\beta_n)=(\eta_1,\eta_2,\dots,\eta_n)P,$$

□□ P □□□□□,□ $\beta_1,\beta_2,\dots,\beta_n$ □ V □□□□□□□.

□□ P □□ j □ Y_j □ β_j □□□□□ $\eta_1,\eta_2,\dots,\eta_n$ □□□□, $j=1,2,\dots,n$. □□ P □□□□□,□□

$$Y_i'Y_j=\delta_{ij},\quad i,j=1,2,\dots,n, \tag{18}$$

□□

$$(\beta_i,\beta_j)=Y_i'Y_j=\delta_{ij},\quad i,j=1,2,\dots,n, \tag{19}$$

□□ $\beta_1,\beta_2,\dots,\beta_n$ □ V □□□□□□□□. X

□□□□□□□□□□□□ V,□□□□□□□□□, V □□□□□□□□□□□,□□□□□□□□□□□□□□□ ?□□□□□□□□,□□□□□□□□□,□□□□□□□□,□□□□□□□□□□□ ?

□□ **7** □ V □ V' □□□□□□□,□□□□ V □ V' □□□□□ σ,□□□□□□ α,β □ V, k □ R, □

$$\sigma(\alpha+\beta)=\sigma(\alpha)+\sigma(\beta),$$
$$\sigma(k\alpha)=k\sigma(\alpha),$$
$$(\sigma(\alpha),\sigma(\beta))=(\alpha,\beta),$$

□□ σ □□□□□□ V □ V' □□□□□□□,□□□ V □ V' □□□□,□□ $V\cong V'$.

□□□ 7 □□,□□□□□ V □ V' □□□□□□ σ □□□□□□□□ V □ V' □□□□□□,□□ σ □□□□□,□□ σ □□□□□□□□□□□□□□,□□□□□□□□□□□,□□ σ □ V □□□□□□□□□□ V' □□□□□□□. □□□□: □ $\eta_1,\eta_2,\dots,\eta_n$ □□□□□□□ V □□□□□□□□,□

$$(\eta_i,\eta_j)=\delta_{ij},\quad i,j=1,2,\dots,n.$$

□□

$$(\sigma(\eta_i),\sigma(\eta_j))=(\eta_i,\eta_j)=\delta_{ij},\quad i,j=1,2,\dots,n.$$

□□ $\sigma(\eta_1),\sigma(\eta_2),\dots,\sigma(\eta_n)$ □□□ V' □□□□(□□□□□□□□□□□□□□),□□□ V' □□□□□□□□□.

□□ **9** □□□□□□□□□□□□□□□□□□□□□□□□.

□□ □ V □ V' □□□□□□□□.

□□□. □□ V □ V' □□□□□□□□□,□□□□□□□□□.

□□□. □ V □ V' □□□□□□□□□: $\eta_1,\eta_2,\dots,\eta_n$ □ $\delta_1,\delta_2,\dots,\delta_n$.

□

$$\sigma: V \to V',$$
$$\alpha = \sum_{i=1}^{n} x_i\eta_i \longmapsto \sum_{i=1}^{n} x_i\delta_i,$$

□ σ □□□□□ V □ V' □□□□□□□. □ $\beta = \sum_{i=1}^{n} y_i\eta_i$, □ $\sigma(\beta) = \sum_{i=1}^{n} y_i\delta_i$.

□□ $\sigma(\alpha) = \sum_{i=1}^{n} x_i\delta_i$, □□

$$(\sigma(\alpha), \sigma(\beta)) = \sum_{i=1}^{n} x_i y_i = (\alpha, \beta).$$

□□ σ □□□□□□ V □ V' □□□□□□□, □□ $V \cong V'$. x

□□□ 8 □□, □□ n □□□□□□ V □□□□□□□□□□□□□□ □□ $\mathbf{R}^n$ □□, □□□□□□□□□:

$$\sigma: V \to \mathbf{R}^n$$
$$\alpha = \sum_{i=1}^{n} x_i\eta_i \longmapsto (x_1, x_2, \dots, x_n),$$

□□ $\eta_1, \eta_2, \dots, \eta_n$ □ V □□□□□□□□.

□□□□, □□□□□□□□□□□□□□□□□□□□□□□□□□□.

□ □ 10.2

1 □ $\mathbf{R}^2$ □, □□□□ $\alpha = (x_1, x_2)$, $\beta = (y_1, y_2)$, □□

$$(\alpha, \beta) \stackrel{\text{def}}{=} x_1y_1 - x_1y_2 - x_2y_1 + 4x_2y_2,$$

□□ (α, β) □□□ $\mathbf{R}^2$ □□□□□□.

2 □□□□□□ $M_n(\mathbf{R})$ □, □□□□□□□□□□

$$f(A, B) \stackrel{\text{def}}{=} \operatorname{tr}(AB'),$$

□□ $f(A, B)$ □□□ $M_n(\mathbf{R})$ □□□□□□.

3 □ $\mathbf{R}^n$ □, □□□□□□□□□□ α, β, □□

$$(\alpha, \beta) \stackrel{\text{def}}{=} \alpha A \beta',$$

□□ A □□□ n □□□□□□. □□: (α, β) □ $\mathbf{R}^n$ □□□□□□.

4 □ $\mathbf{R}^4$ □, □□□□□□□□, □ α □ β □□□□ $\langle\alpha, \beta\rangle$, □□

$$\alpha = (1, -1, 4, 0), \quad \beta = (3, 1, -2, 2).$$

5 □ $\mathbf{R}[x]_3$ □, □□□□□□□□

$$(f(x), g(x)) \stackrel{\text{def}}{=} \int_{-1}^{1} f(x)g(x)\,dx,$$

□ $\mathbf{R}[x]_3$ □□□□□□□□□.

6 [illegible] V [illegible] 3 [illegible], $\alpha_1, \alpha_2, \alpha_3$ [illegible] V [illegible].

$$A = \begin{pmatrix} (\alpha_1,\alpha_1) & (\alpha_1,\alpha_2) & (\alpha_1,\alpha_3) \\ (\alpha_2,\alpha_1) & (\alpha_2,\alpha_2) & (\alpha_2,\alpha_3) \\ (\alpha_3,\alpha_1) & (\alpha_3,\alpha_2) & (\alpha_3,\alpha_3) \end{pmatrix},$$

[illegible] A [illegible] $\alpha_1, \alpha_2, \alpha_3$ [illegible].

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 10 & -2 \\ 1 & -2 & 2 \end{pmatrix},$$

[illegible] V [illegible].

7. [illegible] η_1, η_2, η_3 [illegible] 3 [illegible] V [illegible],

$$\beta_1 = \frac{1}{3}(2\eta_1 - \eta_2 + 2\eta_3),$$

$$\beta_2 = \frac{1}{3}(2\eta_1 + 2\eta_2 - \eta_3),$$

$$\beta_3 = \frac{1}{3}(\eta_1 - 2\eta_2 - 2\eta_3).$$

[illegible]: $\beta_1, \beta_2, \beta_3$ [illegible] V [illegible].

8 [illegible] $\eta_1, \eta_2, \eta_3, \eta_4, \eta_5$ [illegible] 5 [illegible] V [illegible], $V_1 = L(\alpha_1, \alpha_2, \alpha_3)$, [illegible]

$$\alpha_1 = \eta_1 + 2\eta_3 - \eta_5,$$

$$\alpha_2 = \eta_2 - \eta_3 + \eta_4,$$

$$\alpha_3 = -\eta_2 + \eta_3 + \eta_5.$$

(1) [illegible] (α_i, α_j), $1 \leq i, j \leq 3$;

(2) [illegible] V_1 [illegible].

*9. [illegible] R^2 [illegible]

$$(\alpha, \beta) \overset{\text{def}}{=} x_1 y_1 + 2x_2 y_2,$$

[illegible] $\alpha = (x_1, x_2)$, $\beta = (y_1, y_2)$, [illegible] V. [illegible] V [illegible] R^2 [illegible].

§ 3 [illegible], [illegible]

[illegible], [illegible] U [illegible] O [illegible], l [illegible] O [illegible] U [illegible], [illegible] l [illegible] U [illegible], [illegible] l [illegible] U [illegible].

[illegible] V [illegible], [illegible] U [illegible]? [illegible]? [illegible].

□ V□□□□□□□□, V_1□ V□□□□□□□□.□□ V_1□□□□□□□□□□□□(□□ V□□□□□□□□□),□□ V_1□□□□□□□,□ V_1□□□□□□ V□□□□□□.

□□ 1 □ V□□□□□□, S□ V□□□□□□□.□□□ V□□ S□□□□□□□□□□□□□□□□□□□□ S□□□□,□□ $S^{\perp}$.□

$$S^{\perp} \overset{\mathrm{def}}{=} \{\alpha \in V \mid (\alpha, \beta) = 0, \forall \beta \in S\}. \tag{1}$$

□□□□, $S^{\perp}$□ V□□□□□□.

□□□□□□□ V□□□□□□□□□ U,□ V□□□□□□ U□ $U^{\perp}$□□□□□□,□□□□□□□□□□:

□□ 1 □ U□□□□□□ V□□□□□□□□□,□

$$V = U \oplus U^{\perp}. \tag{2}$$

□□ □□ $V = U + U^{\perp}$.□ U□□□□□□□□□ $\varepsilon_1, \ldots, \varepsilon_m$.□□ $\alpha \in V$,□

$$\alpha_1 = (\alpha, \varepsilon_1)\varepsilon_1 + \ldots + (\alpha, \varepsilon_m)\varepsilon_m \in U, \tag{3}$$

$$\alpha_2 = \alpha - \alpha_1. \tag{4}$$

□□□□ $j = 1, \ldots, m$,□

$$(\alpha_2, \varepsilon_j) = (\alpha - \alpha_1, \varepsilon_j) = (\alpha, \varepsilon_j) - \left(\sum_{i=1}^{m} (\alpha, \varepsilon_i)\varepsilon_i, \varepsilon_j\right)$$

$$= (\alpha, \varepsilon_j) - (\alpha, \varepsilon_j) = 0,$$

□□ $\alpha_2 \in U^{\perp}$.□ $\alpha = \alpha_1 + \alpha_2$□□ $V = U + U^{\perp}$.

□□ $U \cap U^{\perp} = 0$.□□ $\gamma \in U \cap U^{\perp}$,□ $(\gamma, \gamma) = 0$,□□□ $\gamma = 0$.

□□□□, $V = U \oplus U^{\perp}$. ×

□□,□□□□□ V□,□ U□□□□□□□□□,□ $U^{\perp}$□□□□□□□□ U□□□□□.□□, $V = U \oplus U^{\perp}$.

□□ 1□□,□□ U□□□□□ V□□□□□□□□,□ $V = U \oplus U^{\perp}$.□□ V□□□□□α□□□□□□

$$\alpha = \alpha_1 + \alpha_2, \quad \alpha_1 \in U, \ \alpha_2 \in U^{\perp}. \tag{5}$$

□□□ V□□□□□□ $P_U: \alpha \longmapsto \alpha_1$.□□□ P_U□□ V□ **U**□□□□□□,□ α_1□□ α□ **U**□□□□□□.□(5)□□□,α_1□ U□α□ U□□□□□□□□□□$\alpha - \alpha_1 \in U^{\perp}$.

□□ 2 □ U□□□□□□□ V□□□□□□,□□ $\alpha \in V$, $\alpha_1 \in U$□α□ U□□□□□□□□□□

$$d(\alpha, \alpha_1) \leqslant d(\alpha, \gamma), \quad \forall \gamma \in U. \tag{6}$$

□□ □□□. □ α_1 □ U □ α □ U □□□□□, □ $\alpha - \alpha_1$ □ $U^{\square}$. □□ " γ □ U, □

$$(\alpha - \alpha_1) \square (\alpha_1 - \gamma).$$

□□□□□□

$$\begin{aligned} |\alpha - \alpha_1|^2 + |\alpha_1 - \gamma|^2 &= |(\alpha - \alpha_1) + (\alpha_1 - \gamma)|^2 \\ &= |\alpha - \gamma|^2. \end{aligned}$$

□□□□, $|\alpha - \alpha_1|$ □ $|\alpha - \gamma|$, □ $d(\alpha, \alpha_1)$ □ $d(\alpha, \gamma)$.

□□□. □(6)□□□. □□ δ □ α □ U □□□□□, □□□□□, $d(\alpha, \delta)$ □ $d(\alpha, \alpha_1)$. □□(6)□□

$$d(\alpha, \delta) = d(\alpha, \alpha_1).$$

□□ $(\alpha - \delta)$ □ $U^{\square}$, $(\delta - \alpha_1)$ □ U, □□

$$|\alpha - \alpha_1|^2 = |(\alpha - \delta) + (\delta - \alpha_1)|^2 = |\alpha - \delta|^2 + |\delta - \alpha_1|^2.$$

□□□□, $|\delta - \alpha_1|^2 = 0$. □□ $\delta = \alpha_1$. X

□□□□□□□□□□.

□□□□□□□□□□□□□□□□□□ $AX = \beta$ □□□□, □□, $A = (a_{ij})_{s \times n}$, $\beta = (b_1, \ldots, b_s)$□. □ A □□□□□□□ $\gamma_1, \gamma_2, \ldots, \gamma_s$. □ $AX = \beta$ □□□, □□□□□□□□□, □□□□□□□□□ $\alpha = (c_1, c_2, \ldots, c_n)$□, □□□ $x_1 = c_1$, $x_2 = c_2$, …, $x_n = c_n$ □, □□□□

$$\sum_{i=1}^{s} [(a_{i1} c_1 + a_{i2} c_2 + \ldots + a_{in} c_n) - b_i]^2 \tag{7}$$

□□□□□, □□□□□ α □□□□□□□□□□□□□. □□□ $AX = \beta$ □□□□□□?

(7)□□□□□□□□, □□□□□□□□□□□□□□ R^s(□□□□□□□□□□)□□□□□□□□□□. □□□□□ i □□□□

$$(a_{i1} c_1 + a_{i2} c_2 + \ldots + a_{in} c_n) - b_i = \gamma_i \alpha - b_i, \quad i = 1, 2, \ldots, s,$$

□□□□□□□

$$\begin{pmatrix} \gamma_1 \alpha - b_1 \\ \gamma_2 \alpha - b_2 \\ \vdots \\ \gamma_s \alpha - b_s \end{pmatrix} = \begin{pmatrix} \gamma_1 \alpha \\ \gamma_2 \alpha \\ \vdots \\ \gamma_s \alpha \end{pmatrix} - \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_s \end{pmatrix} = A\alpha - \beta, \tag{8}$$

□□ α □ $AX = \beta$ □□□□□□□□□ $A\alpha - \beta$ □□□ $|A\alpha - \beta|$ □□, □□□□ X □ R^n, □ $|A\alpha - \beta|$ □ $|AX - \beta|$.

□ A □□□□□ $\alpha_1, \alpha_2, \ldots, \alpha_n$, □ X □ R^n □□□□

$$AX = x_1 \alpha_1 + x_2 \alpha_2 + \ldots + x_n \alpha_n \square \square(\alpha_1, \alpha_2, \ldots, \alpha_n)\square.$$

□ A □□□□□$\alpha_1,\alpha_2,\dots,\alpha_n$□□□ U,□□

$$\begin{aligned}
&\alpha \text{ □ } AX=\beta \text{ □□□□□□}\\
\Leftrightarrow\ & |A\alpha-\beta|\le|AX-\beta|,\ \forall X\in R^n\\
\Leftrightarrow\ & |A\alpha-\beta|\le|\gamma-\beta|,\ \forall \gamma\in U\\
\Leftrightarrow\ & d(A\alpha,\beta)\le d(\gamma,\beta),\ \forall \gamma\in U\\
\Leftrightarrow\ & A\alpha \text{ □ } \beta \text{ □ } U \text{ □□□□□□}\\
\Leftrightarrow\ & \beta-A\alpha\in U^{\perp}\\
\Leftrightarrow\ & (\beta-A\alpha,\alpha_j)=0,\ j=1,2,\dots,n\\
\Leftrightarrow\ & \alpha_j^{\mathrm T}(\beta-A\alpha)=0,\ j=1,2,\dots,n\\
\Leftrightarrow\ & A^{\mathrm T}(\beta-A\alpha)=0\\
\Leftrightarrow\ & A^{\mathrm T}A\alpha=A^{\mathrm T}\beta\\
\Leftrightarrow\ & \alpha \text{ □ } (A^{\mathrm T}A)X=A^{\mathrm T}\beta \text{ □□}.
\end{aligned}$$

□□

$$\operatorname{rank}(A^{\mathrm T}A,\ A^{\mathrm T}\beta)=\operatorname{rank}(A^{\mathrm T}(A,\ B))\le\operatorname{rank}(A^{\mathrm T})=\operatorname{rank}(A^{\mathrm T}A),$$

$$\operatorname{rank}(A^{\mathrm T}A,\ A^{\mathrm T}\beta)\ge\operatorname{rank}(A^{\mathrm T}A),$$

□□

$$\operatorname{rank}(A^{\mathrm T}A,\ A^{\mathrm T}\beta)=\operatorname{rank}(A^{\mathrm T}A),$$

□□□□□□$(A^{\mathrm T}A)X=A^{\mathrm T}\beta$□□□□.□□□□□□□□□□□ $AX=\beta$ □□□□□□□□□□□□□□□□□□$(A^{\mathrm T}A)X=A^{\mathrm T}\beta$□□.

□□ 10 3

1 □ U □□□□□□ R^4(□□□□□□)□□□□□, $U=$□α_1,α_2□,□□

$$\alpha_1=(1,1,2,1),\quad \alpha_2=(1,0,0,-2).$$

□ $U^{\perp}$ □□□□□□□□□.

2 □ V □□□ n □□□□□□□,$\alpha\in V$ □ $\alpha\ne 0$,□□$\alpha^{\perp}$ □□□.

3 □ U □ n □□□□□□□ V □□□□□□,□□:

$$(U^{\perp})^{\perp}=U.$$

4 □□:□□□□□□ R^n(□□□□□□)□□□□□□ U □□□□□□□□□□□□□□□.

5 □ U □ n □□□□□□□ V □□□□□□,□ U □□□□□□□□□□ $\eta_1,\eta_2,\dots,\eta_m$,□ α □ U □□□□□□□ α_1 □

$$\alpha_1=\sum_{i=1}^{m}(\alpha,\eta_i)\eta_i.$$

6 □□□□□□□ R^3(□□□□□□)□,□ $U=$□γ_1,γ_2□,□□ $\gamma_1=(1,2,1)$,$\gamma_2=(1,

0, - 2). □ $\alpha=(1,-3,0)$□ U□□□□□□ α_1.

7. □ U□□□□□□□ V□□□□□□,□ V□ U□□□□□□ P□□□□□□:

$$(P\alpha,\beta)=(\alpha,P\beta),\quad \forall\alpha,\beta\in V.$$

8. $\{\varepsilon_1,\dots,\varepsilon_m\}$□□□□□ V□□□□□□□□□□.□□:□□□□ $\alpha\in V$,□

$$\sum_{i=1}^{m}(\alpha,\varepsilon_i)^2\le|\alpha|^2,$$

□□□□□□□□ $\alpha=\sum_{i=1}^{m}(\alpha,\varepsilon_i)\varepsilon_i$. □□□□□□ Bessel □□□.

9. □□□□□ $R[x]_4$ □,□□□□□□□

$$(f,g)=\int_0^1 f(x)g(x)dx,$$

□ W□□□□□□□□□□□□□□□□□□□□□□□,□ $W^{\perp}$ □□□□□□□.

10. □□□□□ $M_n(R)$, □□□□□□ $(A,B)=\mathrm{tr}(AB')$. □ W□□□□□□□□□□ □□□□,□ $W^{\perp}$ □□ $W^{\perp}$ □□□□□□□□.

§4 □□□□□□□□□

□□□□□□□□□□□ σ □□□□□□□□□□□□,□□ σ □□□□ □□□□□.

□□□□□□□□□□□□□□□□.□□□□□□□□□.

□□ 1 □□□□□ V□□□□□□ A□□□□□□□□□□□□,□

$$(A\alpha,A\beta)=(\alpha,\beta),\quad \forall\alpha,\beta\in V,\tag{1}$$

□□ A□ V□□□□□□□□.

□□□ 1□□□□, V□□□□□□□□□□□□□□□□.

□□ 1 □□□□□ V□□□□□□ A□□□□□□□.

□□ □□ $A(\alpha+\beta)=A\alpha+A\beta$. □□

$$\begin{aligned}
&|A(\alpha+\beta)-(A\alpha+A\beta)|^2\\
=&|A(\alpha+\beta)|^2-2(A(\alpha+\beta),A\alpha+A\beta)+|A\alpha+A\beta|^2\\
=&|\alpha+\beta|^2-2(A(\alpha+\beta),A\alpha)-2(A(\alpha+\beta),A\beta)\\
&+|A\alpha|^2+2(A\alpha,A\beta)+|A\beta|^2\\
=&|\alpha+\beta|^2-2(\alpha+\beta,\alpha)-2(\alpha+\beta,\beta)+|\alpha|^2+2(\alpha,\beta)+|\beta|^2\\
=&|\alpha+\beta|^2-2(\alpha+\beta,\alpha+\beta)+|\alpha+\beta|^2=0,
\end{aligned}$$

□□ $A(\alpha+\beta)-(A\alpha+A\beta)=0$,□ $A(\alpha+\beta)=A\alpha+A\beta$.

□□□□, $A(k\alpha)=kA\alpha$, $\forall\alpha\in V, k\in R$. □□ A□ V□□□□□□□□.

X

[illegible] 2 [illegible] V [illegible] A [illegible]，[illegible] A [illegible].

[illegible] [illegible] $A\alpha = A\beta$，[illegible] $A(\alpha-\beta)=0$. [illegible]

$$|\alpha-\beta| = |A(\alpha-\beta)| = |0| = 0.$$

[illegible] $\alpha-\beta=0$，[illegible] $\alpha=\beta$. [illegible] A [illegible]. [illegible]，A [illegible]. [illegible] A [illegible]，[illegible] A [illegible]. ×

[illegible] 1 [illegible] 2 [illegible] 1 [illegible]

[illegible] 3 [illegible] V [illegible] A [illegible] A [illegible] V [illegible]. ×

[illegible]，[illegible]；[illegible].

[illegible]，[illegible]，[illegible]，[illegible].

[illegible] 4 n [illegible] V [illegible] A [illegible]

(1) A [illegible] V [illegible]

(2) A [illegible] V [illegible] A [illegible].

[illegible] [illegible]. [illegible] A [illegible] V [illegible]，[illegible] A [illegible] V [illegible]，[illegible] A [illegible] V [illegible].

[illegible]. [illegible] V [illegible] A [illegible] V [illegible] $\eta_1, \eta_2, \dots, \eta_n$ [illegible] $\gamma_1, \gamma_2, \dots, \gamma_n$，[illegible] A [illegible]. [illegible] α, β [illegible] V，[illegible] $\alpha = \sum_{i=1}^{n} x_i\eta_i$，$\beta = \sum_{i=1}^{n} y_i\eta_i$，[illegible]

$$A\alpha = \sum_{i=1}^{n} x_i A\eta_i = \sum_{i=1}^{n} x_i\gamma_i,$$

$$A\beta = \sum_{i=1}^{n} y_i A\eta_i = \sum_{i=1}^{n} y_i\gamma_i.$$

[illegible]

$$(A\alpha, A\beta) = \sum_{i=1}^{n} x_i y_i = (\alpha, \beta),$$

[illegible] A [illegible] V [illegible].

[illegible]. [illegible] $\eta_1, \eta_2, \dots, \eta_n$ [illegible] V [illegible]，[illegible] $A(\eta_1, \eta_2, \dots, \eta_s) = (\eta_1, \eta_2, \dots, \eta_n)A$. [illegible] §2 [illegible] 7 [illegible] 8 [illegible]

$A\eta_1, A\eta_2, \dots, A\eta_n$ [illegible] V [illegible]

(2) A [illegible]. ×

□□□□□□□□ V□□□□□□ $\mathcal{A}$□ V□□□□□□□□□□□□ $\mathcal{A}$□□□□□,□□□□□□□□□□□ 1□□ -1,□□□□□□□□□□□ 1□ -1.□□□□□ 1□□□□□□□□□□□(□□□);□□□□□ -1□□□□□□□□□□□.

□□□ 10.3□□ 7□□,□□□□□□□□□□ V□□□□□□ U□□□□□□ $\mathcal{P}$□□□□□□:

$$(\mathcal{P}\alpha,\beta)=(\alpha,\mathcal{P}\beta),\quad \forall\alpha,\beta\in V.$$

□□□□□□,□□□□□□□□.

□□ 2 □□□□□ V□□□□□□ $\mathcal{A}$□□□□

$$(\mathcal{A}\alpha,\beta)=(\alpha,\mathcal{A}\beta),\quad \forall\alpha,\beta\in V,$$

□□ $\mathcal{A}$□□□□□.

□□ 5 n□□□□□□□ V□□□□□□ $\mathcal{A}$□□□□□□□□□ $\mathcal{A}$□ V□□□□□□□□□□□□□□□□□□□□.

□□ □□ V□□□□□□□□ $\alpha_1,\alpha_2,\dots,\alpha_n$.□

$$\mathcal{A}(\alpha_1,\alpha_2,\dots,\alpha_n)=(\alpha_1,\alpha_2,\dots,\alpha_n)A.$$

□ $\mathcal{A}\alpha_i$ □□□□□□ $\alpha_1,\alpha_2,\dots,\alpha_n$ □□□□□□ j □□□□ $a_{ji}=(\mathcal{A}\alpha_i,\alpha_j)$, $i,j=1,2,\dots,n$.□□

$$\begin{aligned}\mathcal{A}\text{□□□□□}&\Leftrightarrow(\mathcal{A}\alpha,\beta)=(\alpha,\mathcal{A}\beta),\ \forall\alpha,\beta\in V\\&\Leftrightarrow(\mathcal{A}\alpha_i,\alpha_j)=(\alpha_i,\mathcal{A}\alpha_j),\ 1\le i,j\le n\\&\Leftrightarrow a_{ji}=a_{ij},\ 1\le i,j\le n\\&\Leftrightarrow A\text{□□□□□}.\end{aligned}$$ ×

□□ 6 □ $\mathcal{A}$□□□□□□ V□□□□□□□□,□□ W□ $\mathcal{A}$□□□□□□,□ $W^{\perp}$□□ $\mathcal{A}$□□□□□□.

□□ □□ $\beta\in W^{\perp}$,□□ $\mathcal{A}\beta\in W^{\perp}$.□□ $\alpha\in W$,□ $\mathcal{A}\alpha\in W$,□□ $(\mathcal{A}\alpha,\beta)=0$.□□

$$(\alpha,\mathcal{A}\beta)=(\mathcal{A}\alpha,\beta)=0,$$

□□ $\mathcal{A}\beta\in W^{\perp}$,□ $W^{\perp}$□ $\mathcal{A}$□□□□□□. ×

□□□□,□□□□□□□□□□□□□□□□□□□,□□□□

□□ 7 □ $\mathcal{A}$□ n□□□□□□□ V□□□□□□□□,□ $\mathcal{A}$□□□□□□□□ V□□□□□□□□□□,□□ $\mathcal{A}$□□□□□□□□□□□□. ×

□ □ 10.4

1 □ V□□□□□□,□□: V□□□□□□ $\mathcal{A}$□□□□□□,□□□□□□□ 1

□ -1.

2 □ V □ n □□□□□□□, η □ V □□□□□□□, □ P □ V □□ η □□□□□□□.
□

$$A = I - 2P,$$

□ A □□□□□□□□ η □□□□□□ (n □□□□□□□□ (n-1) □□□□□□□□□□□ □), □□□□□□□. □□: □□□□□□□□□, □□□□□□□.

3 □ A □ n □□□□□□□ V □□□□□□□□, □□ 1 □ A □□□□□□, A □□ □ 1 □□□□□□ V_1 □□□□ n-1. □□: A □□□□□.

4 □□: □□□□□ V □□□□□□ A □□□□□□□□□ A □□□□□□□□□ □□□□.

5 □ V □□□□□□□□□, A □ V □□□□□□□□, □□: A □ V □□□□□□ □□□□□□□□ A □

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad □ \quad \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix},$$

□□ θ □□□□□.

6 □ α, β □□□□□□□ V □□□□□□□□□□, □□: □□□□□□□□ A, □□ $A\alpha = \beta$.

7 □□: □□□□□□□□□□□□□□ 1 □ -1 □ $\cos\theta \pm i\sin\theta$, θ □□□. □ $0 < \theta < 2\pi$.

8 □□□□□□□□□□□□ 7.

§5 □ □ □

□□□□□□□□□□ V □, □□□□□□□□□? □□□□□□□□□□ □. □□□□□□□□□□□□□□□□□□□? □□□□, □□□□ α □ 0, □

$$(i\alpha, i\alpha) = i^2(\alpha, \alpha) = -(\alpha, \alpha) < 0,$$

$$(i\alpha, i\alpha) > 0,$$

□□□□□. □□□□□□□□□□□□□□□□□□□□:

$$(\alpha, \beta) = \overline{(\beta, \alpha)}.$$

□□□□□□□□□□ (Hermite) □, □□

$$(i\alpha, i\alpha) = i(\alpha, i\alpha) = i(\overline{i\alpha, \alpha}) = i\bar{i}(\alpha, \alpha) = (\alpha, \alpha) > 0,$$

□□□□□□□□.

□□ 1 □□□□□□□□□ V □□□□□□□□□□ (α, β), □□□□□ □□ 4 □□□: ∀ α, β, γ □ V, k □ C, □

1° $(\alpha, \beta) = \overline{(\beta, \alpha)}$ (□□□□□);

2° $(\alpha + \gamma, \beta) = (\alpha, \beta) + (\gamma, \beta)$ (□□□□□);

3° $(k\alpha,\beta)=k(\alpha,\beta)$ (□□□□□);

4° (α,α)□□□□□, $(\alpha,\alpha)=0$□□□□ $\alpha=0$ (□□□),

□□□□□□□□(α,β)□ V□□□□□□.□□□□□ V□□□□□□□□□□,□□ V□□□□.

□ 1 C^n □,□□□□ $X=(x_1,x_2,\dots,x_n)$, $Y=(y_1,y_2,\dots,y_n)$,□□

$$(X,Y)\overset{\text{def}}{=}x_1\bar{y}_1+x_2\bar{y}_2+\dots+x_n\bar{y}_n, \tag{1}$$

□□□□,(X,Y)□ C^n □□□□□□,□□□□□ C^n □□□□□□. C^n □□□□□□□□□,□□□□□□□□.

□ 2 □ □$C[a,b]$□□□□$[a,b]$□□□□□□□□□□□□□□□□,□□

$$(f(x),g(x))\overset{\text{def}}{=}\int_a^b f(x)\overline{g(x)}\,dx, \tag{2}$$

□□□□,$(f(x),g(x))$□ □$C[a,b]$□□□□□□,□□ □$C[a,b]$□□□□□□.

□ 3 □□□ A^* □□□□ A □□□□□□□□□□□□□□(□ A□).□ $M_n(C)$□,□□

$$(A,B)\overset{\text{def}}{=}\operatorname{tr}(AB^*), \tag{3}$$

□□□□,(A,B)□ $M_n(C)$□□□□□□.□□ $M_n(C)$□□□□□□□.

□□□□□□□□,□□□ V□□□□□□□□□□,□□□□□□□□□□□□□□□□□□□□□.

□□ 2 □□□□ (α,α)□□□□ α □□□,□□$|\alpha|$(□□$\|\alpha\|$).

□□$|0|=0$;□ α□0□,$|\alpha|>0$.□□□□:

$$|k\alpha|=|k||\alpha|,\quad \forall\alpha\in V,\ k\in C. \tag{4}$$

□□ 1(Cauchy-Buniakowski □□□) □□□□ V□,□□□□□□ α,β,□

$$|(\alpha,\beta)|\le|\alpha||\beta|, \tag{5}$$

□□□□□□□□ α,β □□□□.

□□ □ α,β □□□□□,□□□□□□□□□□□,□□□

$$|(\alpha,\beta)|=|\alpha||\beta|.$$

□□ α,β □□□□,□□□□□ t,□ $\alpha+t\beta\ne0$.□□

$$0<|\alpha+t\beta|^2=|\alpha|^2+\bar{t}(\alpha,\beta)+t(\beta,\alpha)+t\bar{t}|\beta|^2. \tag{6}$$

□□□,□ $t=-\frac{(\alpha,\beta)}{|\beta|^2}$,□□(6)□□

$$0<|\alpha|^2-\frac{|(\alpha,\beta)|^2}{|\beta|^2}-\frac{|(\alpha,\beta)|^2}{|\beta|^2}+\frac{|(\alpha,\beta)|^2}{|\beta|^2}$$

$$=|\alpha|^2-\frac{|(\alpha,\beta)|^2}{|\beta|^2},$$

[illegible], $|(\alpha,\beta)|<|\alpha||\beta|$. X

[illegible] 3 [illegible] V [illegible], [illegible] α,β [illegible] $\langle\alpha,\beta\rangle$ [illegible]

$$\langle\alpha,\beta\rangle\overset{\text{def}}{=}\arccos\frac{|(\alpha,\beta)|}{|\alpha||\beta|}, \tag{7}$$

[illegible]

$$0\leq\langle\alpha,\beta\rangle\leq\frac{\pi}{2}.$$

[illegible] (7) [illegible], $\langle\alpha,\beta\rangle=\frac{\pi}{2}\Leftrightarrow(\alpha,\beta)=0$.

[illegible] 4 [illegible] V [illegible], [illegible] $(\alpha,\beta)=0$, [illegible] α [illegible] β [illegible], [illegible] $\alpha\perp\beta$.

[illegible], [illegible], [illegible]. [illegible] α,β [illegible]

$$d(\alpha,\beta)\overset{\text{def}}{=}|\alpha-\beta|.$$

[illegible], [illegible] V [illegible], [illegible], [illegible]: [illegible]. [illegible], [illegible], [illegible] V [illegible].

n [illegible] V [illegible], [illegible] $\eta_1,\eta_2,\dots,\eta_n$ [illegible] V [illegible]

$$(\eta_i,\eta_j)=\delta_{ij},\quad i,j=1,2,\dots,n. \tag{8}$$

[illegible] $\eta_1,\eta_2,\dots,\eta_n$, [illegible]. [illegible] α,β [illegible] $\eta_1,\eta_2,\dots,\eta_n$ [illegible] $X=(x_1,x_2,\dots,x_n)$ [illegible], $Y=(y_1,y_2,\dots,y_n)$ [illegible], [illegible]

$$\begin{aligned}(\alpha,\beta)&=\left(\sum_{i=1}^{n}x_i\eta_i,\ \sum_{j=1}^{n}y_j\eta_j\right)\\&=\sum_{i=1}^{n}\sum_{j=1}^{n}x_i\bar{y}_j(\eta_i,\eta_j)\\&=\sum_{i=1}^{n}x_i\bar{y}_i=Y^*X.\end{aligned} \tag{9}$$

[illegible], [illegible]. [illegible] α [illegible] $\eta_1,\eta_2,\dots,\eta_n$ [illegible] $(x_1,x_2,\dots,x_n)$ [illegible], [illegible] $\alpha=\sum_{i=1}^{n}x_i\eta_i$. [illegible] η_j [illegible], [illegible]

$$(\alpha,\eta_j)=\sum_{i=1}^{n}x_i(\eta_i,\eta_j)=x_j,$$

[illegible]

$$\alpha=\sum_{i=1}^{n}(\alpha,\eta_i)\eta_i. \tag{10}$$

(10) [illegible] α [illegible] (Fourier) [illegible], [illegible] (α,η_i) [illegible] α [illegible]

(Fourier)□□.

n□□□□ V□,□ $\eta_1,\eta_2,\dots,\eta_n$ □ V□□□□□□□□,□□□ β_1, $\beta_2,\dots,\beta_n$□□

$$(\beta_1,\beta_2,\dots,\beta_n)=(\eta_1,\eta_2,\dots,\eta_n)P, \tag{11}$$

□ β_i□□□□□□$\eta_1,\eta_2,\dots,\eta_n$□□□□□ P□□ i□ X_i, $i=1,2,\dots,n$,□□

$\beta_1,\beta_2,\dots,\beta_n$ □ V□□□□□□□□

$\Leftrightarrow$ $(\beta_i,\beta_j)=\delta_{ij}$, $i,j=1,2,\dots,n$

$\Leftrightarrow$ $X_j^* X_i=\delta_{ij}$, $i,j=1,2,\dots,n$

$$\Leftrightarrow \begin{pmatrix} X_1^* X_1 & X_1^* X_2 & \dots & X_1^* X_n \\ X_2^* X_1 & X_2^* X_2 & \dots & X_2^* X_n \\ \dots & \dots & & \dots \\ X_n^* X_1 & X_n^* X_2 & \dots & X_n^* X_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} X_1^* \\ X_2^* \\ \dots \\ X_n^* \end{pmatrix} (X_1, X_2, \dots, X_n) = I$$

$$\Leftrightarrow P^* P = I. \tag{12}$$

□□ **5** □□□□ n□□□ P□□□□

$$P^* P = I,$$

□□ P□□□□.

□□□ 5□□

n□□□□ P□□□□ $\Leftrightarrow$ $P^* P=I$

$\Leftrightarrow$ P□□,□ $P^{-1}=P^*$

$\Leftrightarrow$ $PP^*=I$.

□□□□□□□,n□□□□ V□,□□□□□□□□□□□□□□□□□□□□□.□□,□□□□□ $\beta_1,\beta_2,\dots,\beta_n$ □□(11)□.□ P□□□□,□ β_1, $\beta_2,\dots,\beta_n$ □ V□□□□□□□□.

□□□□□□□□□□□,□□□□□□□□□,□□□□□□□□:□□□□□□□□□□□□□□□□□□□□□□□□□□.

□□□□□□□□□□□,□□□□□□□□□□□,□□□□□□□□: n□□□□ V□□□□□□□□□ U□ $U^{\perp}$□□□: $V=U\oplus U^{\perp}$.□□□ n□□□□ V□,□ V□ **U**□□□□□□,□□ **α**□ **U**□□□□□□□□□,□□□□□□□□:α_1□ U□α□ U□□□□□□□□□□

$$d(\alpha,\alpha_1) \;\square\; d(\alpha,\gamma),\ \text{"}\gamma \;\square\; U. \tag{13}$$

□□□□□□□□□□□□□□,□□□□□□□□□.

□□ 6 □□□ V□□□□□□ A□□□□□□□□,□

$$(A\alpha, A\beta) = (\alpha,\beta),\ \text{"}\alpha,\beta \;\square\; V,$$

□□ A□ V□□□□□□□.

□□□□□□□□□□,□□□□:

□□ 2 □□□ V□□□□□□□□□□□□,□□□□□,□□□□□□.

□□,□□□ V□□□□ A□□□□□□□□ A□ V□□□□□□□□□,□□□□□□□□□□□□□□;□□□□□□□□□□□.

□□□□□□□□□□,□□□□

□□ 3 n□□□□ V□□□□□□ A□□□□

□ A□ V□□□□□□□□□□□□□

□ A□ V□□□□□□□□□□□□□□.

□ □ 10 5

1 □□□□ C^3(□□□□□□)□,□

$$\alpha = (1, -1, 1),\ \beta = (1, 0, i),$$

□|α|,|β|,α□β□□□□α,β□.

2 □□□□ C^2(□□□□□□)□,□

$$\alpha_1 = (1, -1),\ \alpha_2 = (1, i),$$

□□ α_1,α_2□□□□□□□□□□ η_1,η_2.

3 □□□□ C^3(□□□□□□)□,□

$$\alpha_1 = (1, -1, 1),\ \alpha_2 = (1, 0, i),$$

□□ α_1,α_2□□□□□□□□□□ β_1,β_2.

4 □□ 1□□□□□□□.

5 □□:□□□□□□□□□□ 1.

6 □□:□□□□□□□□□□ 1.

7. n□□□□ A□□□□

$$A^* = A,$$

□□ A□□□□□(Hermite)□□(□□□□□□),□□ 2□ Hermite□□□□□.

*8 □□□ V□□□□□□ A□□□□

$$(A\alpha,\beta) = (\alpha, A\beta),\ \text{"}\alpha,\beta \;\square\; V,$$

□□ A□□□□□(Hermite)□□(□□□□□□).□□:n□□□□ V□□□□□□ A□

Hermite□□□□□ A□ V□□□□□□□□□□ Hermite□□.

*9 □□:□□□ V□□ Hermite□□ A□□□□□□□□□.

*§6 □□□□□□□□

□□□□□□□□□□□□□□□□□□□□□□,□□□□□□□□□,□□□□□□□□□□□□□□□□□□□□□□□.□□□□□ F□□□□□□ V□,□□□□□□□□□□□□?□□□□□□□□□□□□□□ V,□□□□□□□□,□□□□□□□□□□□□□□□□□□□.□□,□□□□□□(Einstein)□□□□□□“□-□”□□,□□□□□□(Minkowski)□□ V,□□□□□□□ 4□□□□□,□□□□□□□□□□□□□□□□□□ f□□□□,□□□□□.□□□□□□□□□□□□□□□□ V□□□□□□,□□□□□-□□□□□□($\square$,$f(\alpha-\beta,\alpha-\beta)$,□□ α,β□V)□□.

□□ 1 □ F□□□□□ V□□□□□□□□□□□□□□ f,□□ V□□□□□□□,□ f□ V□□□□□□(□□□).□(V,f)□□□□□□□□ f□□□□□,□□ f□□□□□,□(V,f)□□□□□;□□□□□□□□.

□□,□□□□□□□□□,□□ f□

$$f(\alpha,\beta)=x_1y_1+x_2y_2+x_3y_3-c^2t_1t_2, \tag{1}$$

□□ c□□□,$\alpha=(t_1,x_1,x_2,x_3)$,$\beta=(t_2,y_1,y_2,y_3)$.

□□□□□□,□□□□ f□□□□□□,□□□□□□□□□□□□□□□□□□□□□,□□□□□□□□□□.

□□ 2 □□□□□(V,f)□,□□ $f(\alpha,\beta)=0$,□□ α□β□□,□□ α□β.

□□ f□□□□,□□□ $f(\alpha,\beta)=0$□□□ $f(\beta,\alpha)=0$,□□□□ α□β□□,□ β□α□□□.

□□□□□□,□□□□□□□□□□□□□□.□□,□ R^4□,□□ $\alpha=(x_1,x_2,x_3,x_4)$,$\beta=(y_1,y_2,y_3,y_4)$,□

$$f(\alpha,\beta)\overset{\text{def}}{=}x_1y_1-x_2y_2-x_3y_3-x_4y_4, \tag{2}$$

□□□□,f□ R^4□□□□□□□□□□□□□□□□.□□□□□(R^4,f)□,□ $\alpha_1=(1,1,0,0)$,□

$$f(\alpha_1,\alpha_1)=1^2-1^2=0,$$

□□ α_1□□□□□.□□□□□□□□□□□□□□.

□□ 3 □ S□□□□□(V,f)□□□□□,□□□□

$$\{\alpha \square V \mid f(\alpha,\beta)=0,\ \text{"}\beta \square S\},$$

□□ S□□□□,□□ $S^{\square}$.

□□□□,$S^{\square}$□ V□□□□□□.

□□ **1** □(V, f)□□□□□□□□□□□□, W□ V□□□□□□,□

(i) $\dim W + \dim W^{\square} = \dim V$; (3)

(ii) $(W^{\square})^{\square} = W$. (4)

□□ □□□□□□□□ 10 1 □□ 8 □,□□□□□□□□□□,□□□□□□□□.

×

□□ **4** □□□□□□□(V, f)□□□□ $\alpha_1, \dots, \alpha_n$ □□□□□,□□ $\alpha_1, \dots, \alpha_n$□□□□,□ $f(\alpha_i, \alpha_j)=0$,□□□ i□j.

□□□□□□□□□,□□□□□□,□□□□□□□□□□□.□□□□□□□□□□□□□□□□□□□□□□□.□□,□□ 1□(R^4, f)□,□ $\alpha=(1, 1, 0, 0)$,□ α □ 2α □□□□,□□□□□□□□□□.

□□ **2** □□□□ 2□□ F□□□□□□□□□□□(V, f)□□□□□□.

□□ □§ 1□□□ 4,□□ V□□□□$\alpha_1, \dots, \alpha_n$,□□ f□□□□□□□□□ A□□□□□ $\mathrm{diag}\{d_1, \dots, d_n\}$.□□□ $\alpha_1, \dots, \alpha_n$ □□□□,□□□□ V□□□□□□.

×

□□□(V, f)□□□□□□,□□□□□□ A□□□□□,□□ d_i□0, $i=1, \dots, n$.□□ $f(\alpha_i, \alpha_i)$□0,□□□ α_i □□□□□,$i=1, \dots, n$.□□□□□,□(V, f)□□□,□□□□□□□□□□□□□□.

□□ **5** □□□□□□□(V, f)□□□□□□ $\varepsilon_1, \dots, \varepsilon_n$,□□□□□□□,□□

$$f(\varepsilon_i, \varepsilon_i)=0 \square \pm 1,\ i=1, \dots, n.$$

□ F=R□ C□,□□□□□□□(V, f)□□□□□□□,□□□ F=C□,□□□□□□□□□□□□□□□ ε_i,□□

$$f(\varepsilon_i, \varepsilon_i)=0 \square 1.$$

□□□□□□□□□□□□□(V, f),□□□□□□ β□ V□□□□ $\alpha_1, \dots, \alpha_n$□□□□□□□□□□,□□□□

$$\beta = \sum_{i=1}^{n} \frac{f(\beta, \alpha_i)}{f(\alpha_i, \alpha_i)} \alpha_i, \tag{5}$$

□□(5)□□□□□□□.

□□ **3** □ char F□2,□(V, f)□□□□□,□□ W□ V□□□□□□□□□,□

$$V = W \square W^{\square}. \tag{6}$$

□□　□□ α □ W □ $W^{\square}$. □□ W □□□□, □□ rad $W=0$, □□ $\alpha=0$. □□ W □ $W^{\square}=0$.

□□ $V = W + W^{\square}$. □ W □□□□□□□□□□□□□□□□ $\alpha_1, \ldots, \alpha_m$, □□ V □□□□□ β, □

$$\beta_1 = \square_{i=1}^{m} \frac{f(\beta, \alpha_i)}{f(\alpha_i, \alpha_t)} \alpha_i \square W, \quad \beta_2 = \beta - \beta_1.$$

□□
$$f(\beta_2, \alpha_j) = f(\beta, \alpha_j) - \square_{i=1}^{m} \frac{f(\beta, \alpha_i)}{f(\alpha_i, \alpha_i)} f(\alpha_i, \alpha_j)$$
$$= f(\beta, \alpha_j) - f(\beta, \alpha_j) = 0, \quad j = 1, \ldots, m,$$

□□ β_2 □ $W^{\square}$, □□

$$\beta = \beta_1 + \beta_2 \square W + W^{\square},$$

□□ $V = W \square W^{\square}$. X

□□ **6**　□□□ 2 □□ F □□□□ V □□□□□□□□□□□□□□□□□ f, □□ V □□□□□□, □ (V, f) □□, □ f □ V □□□□□□ (□□□□); □□ f □□□□□, □□ (V, f) □□□□; □□□□□□□□.

□□, R^2 □, □□ $\alpha = (x_1, x_2)$, $\beta = (y_1, y_2)$, □

$$f(\alpha, \beta) \overset{\text{def}}{\square} x_1 y_2 - x_2 y_1, \tag{7}$$

□□□□, f □ R^2 □□□□□□□□□□□□□□□. □□ (R^2, f) □□□□□□□.

□□□□□□□, □□□□□□□□□, □□□□□ □□ □□□□□, □□□□□□□□□□.

□ § 1 □□□ 20 □□, □□□□□□□□□□□□□□□□□□.

□ § 1 □□□ 5 □□, n □□□□ (V, f) □, □□ V □□□□ $\varepsilon_1, \varepsilon_{-1}, \ldots, \varepsilon_r, \varepsilon_{-r}, \eta_1, \ldots, \eta_s$, □□ f □□□□□□□□□□ A □

$$\mathrm{diag}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 0, \ldots, 0 \right),$$

□
$$\begin{aligned} & f(\varepsilon_i, \varepsilon_{-i}) = 1, \quad i = 1, \ldots, r, \\ & f(\varepsilon_i, \varepsilon_j) = 0, \quad i + j \square 0, \\ & f(\varepsilon_i, \eta_k) = 0, \quad i = \pm 1, \ldots, \pm r, \\ & f(\eta_j, \eta_k) = 0, \quad j, k = 1, \ldots, s, \end{aligned} \tag{8}$$

□□□□□ (V, f) □□□.

□□□□ $\varepsilon_1, \ldots, \varepsilon_r, \varepsilon_{-1}, \ldots, \varepsilon_{-r}, \eta_1, \ldots, \eta_s$ □□□□□. □□□□, f □□□□□□□□□

$$\begin{pmatrix} 0 & I_r & 0 \\ -I_r & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

□□ 4 □(V, f)□ n □□□□,□□ W□ V□□□□□□,□

$$V = W \oplus W^{\perp}. \tag{9}$$

□□ □□ $\alpha \in W \cap W^{\perp}$,□□ W□□□□,□□ $\mathrm{rad}\, W = 0$,□□ $\alpha = 0$.□□ $W \cap W^{\perp} = 0$.

□ W□□□□□□$\varepsilon_1, \varepsilon_{-1}, \dots, \varepsilon_m, \varepsilon_{-m}$.□□□□□ V□□□□:$\varepsilon_1, \varepsilon_{-1}, \dots, \varepsilon_m, \varepsilon_{-m}, \beta_1, \dots, \beta_t$,□

$$\beta_j' = \beta_j - \sum_{i=1}^{m} f(\beta_j, \varepsilon_{-i})\varepsilon_i + \sum_{i=1}^{m} f(\beta_j, \varepsilon_i)\varepsilon_{-i}, \quad j = 1, \dots, t,$$

□

$$f(\beta_j', \varepsilon_k) = f(\beta_j, \varepsilon_k) - \sum_{i=1}^{m} f(\beta_j, \varepsilon_{-i}) f(\varepsilon_i, \varepsilon_k) + \sum_{i=1}^{m} f(\beta_j, \varepsilon_i) f(\varepsilon_{-i}, \varepsilon_k)$$

$$= f(\beta_j, \varepsilon_k) + f(\beta_j, \varepsilon_k) f(\varepsilon_{-k}, \varepsilon_k) = 0, \quad k = 1, \dots, m,$$

$$f(\beta_j', \varepsilon_{-k}) = f(\beta_j, \varepsilon_{-k}) - f(\beta_j, \varepsilon_{-k}) f(\varepsilon_k, \varepsilon_{-k}) = 0, \quad k = 1, \dots, m$$

□□ $\varepsilon_1, \varepsilon_{-1}, \dots, \varepsilon_m, \varepsilon_{-m}, \beta_1', \dots, \beta_t'$□□ V□□□□.

□

$$U = L(\beta_1', \dots, \beta_t').$$

□ $V = W \oplus U$,□□□□□□□□□□□ $\beta_j' \in W^{\perp}$, $j = 1, \dots, t$.□□ $U \subset W^{\perp}$.□□□□,□□ $\alpha \in W^{\perp}$,□

$$\alpha = \alpha_1 + \alpha_2, \quad \alpha_1 \in W, \quad \alpha_2 \in U,$$

□□ $\beta \in W$,□

$$0 = f(\alpha, \beta) = f(\alpha_1, \beta) + f(\alpha_2, \beta) = f(\alpha_1, \beta),$$

□□ $\alpha_1 \in W^{\perp}$.□ $\alpha_1 \in W$,□□ $\alpha_1 = 0$.□□ $\alpha \in U$.□□ $U = W^{\perp}$.□□□ $V = W \oplus W^{\perp}$.

*□ □ 10 6

1 R^2□,□□ $\alpha = (x_1, x_2)$, $\beta = (y_1, y_2)$,□□

$$f(\alpha, \beta) = x_1 y_1 - x_2 y_2.$$

(1) □□:(R^2, f)□□□□□□□□□□;

(2) □ $\varepsilon_1 = (1, 0)$, $\varepsilon_2 = (0, 1)$,□□:$\varepsilon_1, \varepsilon_2$□$(R^2, f)$□□□□□□□□;

(3) □ f□□$\varepsilon_1, \varepsilon_2$□□□□□□ A;

(4) □

$$T = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

□ T□ R^2 □□□□□□□□,□□□ $\varepsilon_1,\varepsilon_2$ □□□□□ T. □□: T□□□□□(R^2, f)□□□□□□□□(□ F□□□□□□□□□□ V□□□□□□□ T□□□□□□□,□□ T□ V□□□□□□□□),□□□□□□□□ 2 ± 1;

(5) □ T□□□□□□□,□□□□ T□□□□□□□□□□.

2 □□□□□□(5).

3 □(V, f)□□□□□□□□□, V□□□□□□□ B□□□□□□□□□,□□ B□ V□□□□□. □□: V□□□□□□ B□□□□□□□□ B□ V□□□ $\varepsilon_1,\dots,\varepsilon_r,\varepsilon_{-1},\dots,\varepsilon_{-r}$□□□□ B□□ $B'AB=A$,□□

$$A = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}.$$

□□ $B'AB=A$□□□ B□□□□□.

□□□□□□□

□ 7□ □□□□

□ □ 7.1

1 $f(x)$□ $g(x)$□□□□□.

2 $f(x)$□ $g(x)$□□□□□ 0.

3 $f(x)+g(x)$□□□□□□□ 3,□□□□ 3.□□,□ $f(x)=x^3+1$, $g(x)=-x^3+x^2-1$,□ $f(x)+g(x)$□□□□ 2.

4 □□: $f(x)$□□□□□□□□□□ $g(x)$□ $K[x]$,□□, $f(x)g(x)=1$.□□□□□□□□□□□□□.

*5 □□:□□□□.

6 □□:□

$$D=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

□ $$A=I+bD+b^2D^2+\cdots+b^{n-1}D^{n-1}.$$

□□□□□□□□□□□□□□□□.

7. □□:□ $A=(a_{ij})$,□□□□□□□□

$$|\lambda I-A|=\sum_{j_1j_2\cdots j_n}(-1)^{\tau(j_1j_2\cdots j_n)}(\lambda\delta_{1j_1}-a_{1j_1})\cdots(\lambda\delta_{nj_n}-a_{nj_n}).$$

□□□□□□□□□□□□□□(λ□ $k^{-1}\lambda$□□).

*8 □□: $|\lambda I+A||\lambda I-A|=|\lambda^2 I-A^2|$.□ $A^2=(b_{ij})$.

□ $$|\lambda^2 I-A^2|=\sum_{j_1j_2\cdots j_n}(-1)^{\tau(j_1j_2\cdots j_n)}(\lambda^2\delta_{1j_1}-b_{1j_1})\cdots(\lambda^2\delta_{nj_n}-b_{nj_n}).$$

□□□□ 7□□□□□□□□□□(□ $k=-1$),□□□□ $K[\lambda^2]$□□□□□□□□□□□□□(λ^2□ λ□□).

□ □ 7.2

1 □□□□□□□□□□□□.

2 □□□□□□□□□□□□.

3 (1) □□□ x^2+2x-4, □□□ $-20x+19$;

(2) □□□ $\frac{1}{3}x^2+\frac{4}{9}x-\frac{2}{27}$, □□□ $-\frac{80}{27}x+\frac{85}{27}$.

4 $g(x)\mid f(x)$□□□□ $a_1=3$□ $a_0=-1$.

5 (1) □□□ $3x^3+12x^2+43x+174$, □□□ 695;

(2) □□□ $5x^2-10x+17$, □□□ -30.

*6 □□□□□□□□□(1)□(4)□□□.

□□ 7.3

1 (1) $(f(x), g(x))=x+3$;

(2) $(f(x), g(x))=x-1$.

2 □□ $f(x)$□ $g(x)$□□□□□□ $c(x)$□□□ $d(x)$.

3 □□□□ 3□□ 2□□□□.

4 □ $f(x)=f_1(x)(f(x), g(x))$, $g(x)=g_1(x)(f(x), g(x))$. □□□□ 3□□□ 4□□□, $(f_1(x), g_1(x))=1$.

5 □□□□ 4.

6 □□□□ 4□□□, $(f, f+g)=1$. □□□, $(g, f+g)=1$. □□□□□□ 3□□□, $(fg, f+g)=1$.

7. □□ $af+bg$□ $cf+dg$□□□□□□□□□□□□ f□ g□□□□□□□□□, □□□□□ 1.

8 □□□ 4□□□□□□□□□□□□□.

9 □□□. □ $(f(x), g(x))=d(x)$, □□ $\deg d(x)>1$. □ $f(x)=f_1(x)d(x)$, $g(x)=g_1(x)d(x)$, □□□□□.

□□□, □□□□.

10 (1) □ $f(x)$□ $g(x)$□□□□□□□ $d(x)$, □ $f(x)=f_1(x)d(x)$, $g(x)=g_1(x)d(x)$. □ $m(x)=f_1(x)g_1(x)d(x)$, □□ $m(x)$□ $f(x)$□ $g(x)$□□□□□□□□.

(2) □(1)□□□□□□, $m(x)=\frac{f(x)g(x)}{d(x)}$, □□□□□□□□.

*11 □□□ 3□□□□□□□□□□□□□□□: □□ α □□ $f(A)X=0$□□□□ W_1□ $g(A)X=0$□□□□ W_2□□, □ α □ $d(A)X=0$□□. □ $f(x)=f_1(x)d(x)$, $g(x)=g_1(x)d(x)$, □□□□□□□□□□□□□□□: □□ γ □ $d(A)X=0$□□, □ γ □ $f(A)X=0$□□, □□ γ □ $g(A)X=0$□□.

*12 □□□□ 4□□□□□□□□□□□□□□□□: $f(A)X=0$□□□□□□□□□ $f_1(A)X=0$□□□□□ $f_2(A)X=0$□□□□□. □□□□□□□ 11□□□□□□.

□□ 7.4

1 □□□□□.

2 (1) □□□□□:

$$x^4+1=\left(x+\frac{2}{2}-\frac{2}{2}i\right)\left(x-\frac{2}{2}-\frac{2}{2}i\right)\left(x+\frac{2}{2}+\frac{2}{2}i\right)\left(x-\frac{2}{2}+\frac{2}{2}i\right);$$

□□□□□:

$$x^4+1=(x^2+2x+1)(x^2-2x+1);$$

□□□□□□: x^4+1□□□.

(2) □□□□□:

$$x^4+4=[x+(1-i)][x-(1-i)][x+(1+i)][x-(1+i)];$$

□□□□□:

$$x^4+4=(x^2+2x+2)(x^2-2x+2);$$

□□□□□□:

$$x^4+4=(x^2+2x+2)(x^2-2x+2).$$

3 □□□□□□□.□□□□□ $f(x)$□ $g(x)$□□□□□□□ $f^2(x)$□ $g^2(x)$□□□□□□.□ $g^2(x)\mid f^2(x)$□□□ $g^2(x)$□□□□□□□□□ $f^2(x)$□□□□□□□□,□□□□□ $g(x)\mid f(x)$.

4 □□□□,□□□□□□ 3.

*5 □□□.□□□□□ 1.□□□.□□□□,□□ $f(x)$□□□□□□□□□□□□□,□□□□□□ $g(x)$□ $K[x]$□□,$(f(x), g(x))$□1□□□□□□ m□□ $f(x)8\, g^m(x)$.□□□

$$f(x)=cp_1^{r_1}(x)\,p_2^{r_2}(x)\dots p_s^{r_s}(x).$$

□□□, s□2.□ $g(x)=p_2(x)$.

*6 □□□.□□□5□□□□.□□□.□□□□.□□ $f(x)$□□□□□□□□□□□□□,□ $f(x)$□□□□□□□

$$f(x)=cp_1^{r_1}(x)\,p_2^{r_2}(x)\dots p_s^{r_s}(x),$$

□□ s□2,□ $g(x)=cp_1^{r_1}(x)$, $h(x)=p_2^{r_2}(x)\dots p_s^{r_s}(x)$.

7. □□ g_1□ g_2□□□□□□□□□ fg_1□ g_2□□□□.□□,□ $c(x)$□ fg_1□ g_2□□□□,□□§3□□□3□□□4,□□ $c(x)$□□ g_1□ g_2□□□□,□□□§3□□□1□□□□□.

□ □ 7.5

1 (1) □□□□ $x-2$;

(2) □□□□□.

2 $f(x)=x^3+2ax+b$□□□□□□□□□□□□

$$32a^3+27b^2=0.$$

3 □□ $f(x)=x^k+1$(k□2),□ f□$(x)=kx^{k-1}$.

4 □ $p(x)$□ $f(x)$□ m□□□,□ $p(x)$□ f□(x)□ $m-1$□□□.□□□□□□, $m-1=k-1$,□□ $m=k$.

5 □□□.□□□1□□.□□□.□□□4□□□□.

6 $g(x) = x^2 - 1$, $f(x) = (x+1)(x-1)^4$.

*7. □□□□□.□□□.□□□□ $f(x)$□□□□□□□□□□□□□□,

$$f(x) = g(x)(f(x), f'(x)),$$

□□ $g(x)$□ $f(x)$□□□□□□□□□□□□,□ $g(x)$□□□□□.□□ $f'(x) | f(x)$,□□ $(f(x), f'(x)) = cf'(x)$.□□ $f(x) = cg(x) f'(x)$.□□□□□□□□□□□□□□□ $\deg g(x) = 1$,□□ $g(x) = a(x-b)$, a, b□ K.□ $g(x)$□ $f(x)$□□□□□□□□□□□□□□□□□□□.

□□ 7.6

1 -2□ $f(x)$□ 3□□.

2 $f(x)$□□□□□□□ $a=4$□ $a = -\frac{17}{27}$.

□ $a=4$□, 2□ $f(x)$□□□□;□ $a = -\frac{17}{27}$□, $\frac{1}{3}$□ $f(x)$□□□□.

3 $f(x)$□ $g(x)$□□□□□□□:2.

4 3□ $f(x)$□□□□□□□□ $a=4$□ $b=3$.

5 □□□□□□□□□□□,□□□□□□□□□□□□□.

6 □□□□□□,□□□□□□□□□□□□□ x□□□ ω, ω^2□□,□□ $\omega = \frac{-1+\sqrt{3}i}{2}$ □□□□□ $f_1(1)$□ $f_2(1)$□□□□□.

7. □□□□□□,□□□□□□□□□□□□□□□□:□ c□ $f(x)$□□□□□,□ c^m□□ $f(x)$□□□□□.

8 □ $\xi = e^{i\frac{2\pi}{n}}$,□ $1, \xi, \xi^2, \ldots, \xi^{n-1}$□□ n□□□□,□□□□□□□,□□

$$x^n - 1 = (x-1)(x-\xi)(x-\xi^2)\ldots(x-\xi^{n-1}).$$

*9 (1) □ $m(x)$□ J_a□□□□□□□□□□□ 1□□□□,□□ $f(x)$□ J_a,□ $m(x)$□□ $f(x)$,□□□□□□□□□□□□□□.

(2) □□□□,□□ §4□□□□□□□□□ 3,□□□□□□□□□□□□□□.

10 □□ K□□□□□□□ $p(x)$□ $K[x]$□□□□□,□□□ $C[x]$□, $p(x)$□□□□□□.□□ $p(x)$□□□□□□□□□.

11 □□□□□□□□ 176□□□□ 2, A□□□□□□□

$$f(\lambda) = |\lambda I - A| = \lambda^n - \mathrm{tr}(A) + \ldots + (-1)^n |A|.$$

□□□ Vieta□□.

12 □□ $p(x)$□ $g(x)$□ $K[x]$□□□□.□□□□□□□□□□□□□□□□□□.

□□ 7.7

1 □□□□□□□□□□□□□□□□□□.

2 □ $\xi = e^{i\frac{2\pi}{n}}$,□ 1□ $j < n$□, ξ^j□ ξ^{n-j}□□□□□□□.□□□ $n = 2m+1$□,

$$x^n-1=(x-1)\left(x^2-2\cos\frac{2\pi}{n}x+1\right)\left(x^2-2\cos\frac{4\pi}{n}x+1\right)\cdots\left(x^2-2\cos\frac{2m\pi}{n}x+1\right);$$

□ $n=2m$ □,

$$x^n-1=(x-1)(x+1)\left(x^2-2\cos\frac{2\pi}{n}x+1\right)\cdots\left(x^2-2\cos\frac{2(m-1)\pi}{n}x+1\right).$$

3 (1) □ A □□□□□,□ $f(\lambda)$ □□□□□ λ_0 □ A □□□□,□□□□ n □□□□ ξ □□ $A\xi=\lambda_0\xi$.□□□□□□ z □□□□□□□□□□ $\bar{z}=-z$.□□□□□□ $\bar{\lambda}_0+\lambda_0=0$. □□□□□□□□□□□□□ 183 □□□□ 1 □□□.

(2) □□(1)□□□□□□□□□□□□□□□□□□□□□□□.

*4 □□□□.□□ A □ B □□□□□□□,□□□□□□□□ n □□□□□ U □□, $U^{-1}AU=B$.□ $U=P+iQ$,□□ P,Q □□□□□.□ P,Q □□,□□□□□□□□□□ S.□□□□□□ t,□□□□□ $|P+tQ|$,□□ $g(t)$,□□ t □□□ n □□□□□.□□□□ □□□□□□□□ S □□□□□.

□ □ 7.8

1 (1) $\frac{1}{2}$; (2) $-\frac{1}{2}$, 3.

2 (1) □□□;(2) □□□;

(3) x □ $x-1$ □□,□□□ Eisenstein □□□□□,□□□;

(4) x □ $x+1$ □□,□□□;

(5) x □ $x-1$ □□,□□□;

(6) □□□□□,□□ 3 □□□□,□□□□□;

(7) □□□□ $-\frac{1}{2}$,□□□□;

(8) □□□□□;□□□□□□□□□,□□□□□□,□□□□□.

3 □ $c=\sqrt[n]{p_1p_2\cdots p_n}$,□□ $p_1,p_2,\cdots,p_n$ □□□□□□□□.□ $c^n=p_1p_2\cdots p_n$.□□ c □□□□,□□□□ $x^n-p_1p_2\cdots p_n$ □□□□□□ c,□□□□□□□□□.□□ $n>1$,□□ $x^n-p_1p_2\cdots p_n$ □ Q □□□.□□□ Eisenstein □□□□□, $x^n-p_1p_2\cdots p_n$ □ Q □□□□, □□.

4 $f(1)=a_n+\cdots+a_1+a_0$ □□□,□□ 1 □□ $f(x)$ □□.□□ -1 □ $f(x)$ □□,□ $x+1$ □ $f(x)$ □ $Q[x]$ □□□□.□ $f(x)=rf_1(x)$,□□ $f_1(x)$ □□□□□□, r □ Z^*,□ $x+1$ □□ $f_1(x)$ □ $Q[x]$ □□□□.□□□ 3 □□□□□□, $f_1(x)=(x+1)h(x)$,□□ $h(x)$ □□□□□□.□□ $f(x)=r(x+1)h(x)$.□□ $f(1)=2rh(1)$,□□ $f(1)$ □□□□ □,□□ -1 □□ $f(x)$ □□.

□□ -1 □□ $f(x)$ □□□□□□□□:□□ -1 □ $f(x)$ □□,□ $a_n(-1)^n+a_{n-1}(-1)^{n-1}+\cdots+a_1(-1)+a_0=0$.□□

$$a_n+a_{n-1}+\cdots+a_1+a_0=\begin{cases}2(a_1+a_3+\cdots+a_{2m-1}), & n=2m\\ 2(a_1+a_3+\cdots+a_{2m+1}), & n=2m+1.\end{cases}$$

□□ $a_n+a_{n-1}+\dots+a_1+a_0$ □□□□□.

5 □□ $f(x)$□□□□ c.□□ $f(x)$□□□□□□ 1,□□ $f(x)$□□□□□□,□□□□□ 4 □,c□□□□.□□ $x-c$□ $f(x)$□□□.□□□ 3□□□□□□, $f(x)=(x-c)h(x)$,□□ $h(x)$□□□□□□,□□

$$f(0)=-ch(0),\quad f(1)=(1-c)h(1).$$

□□ $-c$□ $(-c+1)$□□□□□□□,□□ $f(0)$□ $f(1)$□□□□□□□,□□.

6 □$(a+b)c$□□□□□□, $f(0)$□ $f(1)$□□□□,□□□□ 5□□□□.

7. □ $n=1$,□□ $f(x)$□□□.□□□ $n>1$,□□ $f(x)$□ Q□□□,□□□□ 2□,□□ $f_1(x), f_2(x)$□ $Z[x]$□□

$$f(x)=f_1(x)f_2(x),\quad \deg f_i(x)<\deg f(x),\ i=1,2.$$

□□

$$f_1(a_i)f_2(a_i)=f(a_i)=1,\quad i=1,2,\dots,n,$$

□□ $f_1(a_i)$□ $f_2(a_i)$□□ 1□□□ -1,□□□□

$$f_1(a_i)-f_2(a_i)=0,\quad i=1,2,\dots,n.$$

□□

$$\deg(f_1(x)-f_2(x))\ \square\ \max\{\deg f_1(x),\deg f_2(x)\}<\deg f(x)=n,$$

□□ $f_1(x)-f_2(x)=0$,□□ $f_1(x)=f_2(x)$.□□ $f(x)=f_1^2(x)$.□□□ $\deg f(x)$□□□.□□,□□ n□□□,□ $f(x)$□ Q□□□□.

$n=2$□,□ $f(x)=(x-1)(x+1)+1=x^2$□ Q□□□.

$n=4$□,□ $f(x)=(x-1)x(x+1)(x+2)+1$,□

$$\begin{aligned}f(x)&=x^4+2x^3-x^2-2x+1\\&=(x^2+x-1)^2,\end{aligned}$$

□□ $f(x)$□ Q□□□.

*8 □□ $f(x)$□ Q□□□,□□□ $f_1(x), f_2(x)$□ $Z[x]$□□

$$f(x)=f_1(x)f_2(x),\quad \deg f_i(x)<\deg f(x),\quad i=1,2.$$

□□ $$-1=f(a_i)=f_1(a_i)f_2(a_i),\quad i=1,2,\dots,n.$$

□□ $$f_1(a_i)+f_2(a_i)=0,\quad i=1,2,\dots,n.$$

□□□□, $f_1(x)+f_2(x)=0$,□□ $f(x)=-f_1^2(x)$.□□ $f(x)$□□□□□□ 1,□□ $f_1(x)$□ $f_2(x)$□□□□□□□ 1□□□ -1,□□ $-f_1^2(x)$□□□□□□ -1,□□.

*9 □□ $f(x)$□ Q□□□,□□□ $f_1(x), f_2(x)$□ $Z[x]$□□

$$f(x)=f_1(x)f_2(x),\quad \deg f_i(x)<\deg f(x),\quad i=1,2.$$

□□ $$1=f(a_i)=f_1(a_i)f_2(a_i),\quad i=1,2,\dots,n.$$

□□ $f_1(a_i)$□ $f_2(a_i)$□□ 1□□□ -1,□□ $f(x)$□□□□,□□ $f_1(x), f_2(x)$□□□□□□,□□ $f_1(a_i)$□□ 1□□□ -1, $i=1,2,\dots,n$.□□□ $f_1(a_1)=f_1(a_2)=\dots=f_1(a_n)=1$.□□□□

$$f_2(a_1)=\dots=f_2(a_n)=1.$$

□□ 1 $f_1(x)$□ $f_2(x)$□□□□□□□□□□□□ n,□□□ $\deg f_1(x)<n$.□□

$f_1(a_i)=1, i=1,2,\dots,n$, □□ $f_1(x)-1=0$, □□□□ $\deg f_1(x)=0$, $\deg f_2(x)=\deg f(x)$, □□.

□□ 2 $\deg f_1(x)=\deg f_2(x)=n$.

□□ $f_1(a_i)-1=0, i=1,2,\dots,n$. □□

$$f_1(x)-1=(x-a_1)(x-a_2)\cdots(x-a_n).$$

□□,
$$f_2(x)-1=(x-a_1)(x-a_2)\cdots(x-a_n).$$

□□
$$f(x)=f_1(x)f_2(x)=\prod_{i=1}^{n}(x-a_i)^2+2\prod_{i=1}^{n}(x-a_i)+1.$$

□□□□ $2\prod_{i=1}^{n}(x-a_i)=0$, □□.

□□ 7.9

1 (1) $f(x_1,x_2,x_3,x_4)=-x_1^3x_2+2x_2^4x_3x_4+5x_2x_3x_4+x_3^4x_4$;

(2) $f(x_1,x_2,x_3,x_4)=x_1^3-5x_1^2x_3x_4^2+3x_1x_2^2x_4-2x_2^3x_3+x_3^2$.

2 $f(x_1,x_2,x_3)=(x_1+x_2+x_3)(x_1^2+x_2^2+x_3^2-x_1x_2-x_1x_3-x_2x_3)$.

3 □□ $f(x_1,x_2,\dots,x_n)$□0, □□ $g(x_1,x_2,\dots,x_n)$□0, □□ $f(x_1,x_2,\dots,x_n)g(x_1,x_2,\dots,x_n)$□0, □□ fg□□□□□. □□□ $c_1,c_2,\dots,c_n$□ K, □□ $(fg)(c_1,c_2,\dots,c_n)$□0. □□ $f(c_1,c_2,\dots,c_n)g(c_1,c_2,\dots,c_n)$□0. □□

$$f(c_1,c_2,\dots,c_n)□0, □ g(c_1,c_2,\dots,c_n)□0.$$

□□□□□□□□. □□ $f(x_1,x_2,\dots,x_n)=0$.

*4 (1) □□□□□. □□□. □□□□□□.

(2) □□□□,□□□□□□□□.

□□ 7.10

1 3□□□□□□□□□□□ $x_1^3x_2^2$ □□□, □□□□

$$x_1^3x_3^2,\quad x_2^3x_1^2,\quad x_2^3x_3^2,\quad x_3^3x_1^2,\quad x_3^3x_2^2.$$

□□□□ $f(x_1,x_2,x_3)$□ 3□□□□□□.

2 $x_1^3x_2+x_1^3x_3+x_2^3x_1+x_2^3x_3+x_3^3x_1+x_3^3x_2$.

3 (1) $\sigma_1^2\sigma_2-2\sigma_2^2-\sigma_1\sigma_3$;

(2) $\sigma_1^4-4\sigma_1^2\sigma_2+2\sigma_2^2+4\sigma_1\sigma_3$;

(3) $\sigma_1^3\sigma_3+\sigma_2^3-6\sigma_1\sigma_2\sigma_3+8\sigma_3^2$.

4 (1) $\sigma_1^3-3\sigma_1\sigma_2+3\sigma_3$;

(2) $n=3$ □, $\sigma_2\sigma_3$; $n=4$ □, $\sigma_2\sigma_3-3\sigma_1\sigma_4$;

n□5 □, $\sigma_2\sigma_3-3\sigma_1\sigma_4+5\sigma_5$.

5 $s_2=\sigma_1^2-2\sigma_2$, $s_3=\sigma_1^3-3\sigma_1\sigma_2+3\sigma_3$,

$s_4=\sigma_1^4-4\sigma_1^2\sigma_2+4\sigma_1\sigma_3+2\sigma_2^2$.

6 $D(f)=0$ □, $f(x)$□ C □□□□. □□ $\deg f(x)=3$, □□ $f(x)$□□□□□□□,

$f(x)$□□□□□□□□□□□□□,□ $f(x)$□□□□,□□.□□ $f(x)$□□□□□□□□ □□.

□□ $f(x)$□□□□□□□□ r_1, r_2, r_3,□

$$D(f)=(r_2-r_1)^2(r_3-r_1)^2(r_3-r_2)^2>0.$$

□□ $f(x)$□□□□□ r,□□□□□ $c, \bar{c}$,□ $c=a+bi$,□

$$D(f)=(c-r)^2(\bar{c}-r)^2(\bar{c}-c)^2=|c-r|^4(-2bi)^2$$
$$=-4b^2|c-r|^4<0.$$

□□,□ $D(f)>0$□,$f(x)$□□□□□□□,□□□□□;□ $D(f)<0$□,$f(x)$□□□□□,□□□□□□.

7. □ $f(x)$□□□□ $c_1, c_2, \ldots, c_n$;□ $g(x)$□□□□ $a, c_1, c_2, \ldots, c_n$.□□

$$D(g)=\prod_{1\leqslant j<i\leqslant n}(c_i-c_j)^2\prod_{k=1}^{n}(c_k-a)^2$$
$$=D(f)\prod_{k=1}^{n}(a-c_k)^2=D(f)f(a)^2.$$

8 □□ Vieta□□□

$$\sigma_1(c_1, c_2, \ldots, c_n)=\sigma_2(c_1, c_2, \ldots, c_n)=\ldots=\sigma_{n-1}(c_1, c_2, \ldots, c_n)=0,$$
$$\sigma_n(c_1, c_2, \ldots, c_n)=(-1)^n a,$$

□□ $c_1, c_2, \ldots, c_n$ □ $f(x)=x^n+a$□ n□□□.□□□□□□□□□;□□□□□□□(27)□□□

$$D(f)=(-1)^{\frac{1}{2}(n-1)(n-2)}a^{n-1}n^n.$$

9 $D(f)=-4a_1^3a_3+a_1^2a_2^2+18a_1a_2a_3-4a_2^3-27a_3^2.$

□ □ 7.11

1 □ a□ F□ a□ 0.□□ $ab=0$,□□□□□ a^{-1}□,$b=0$,□□ F□□□□□□□□.

2 Z_7□,$\bar{1}^{-1}=\bar{1}$, $\bar{2}^{-1}=\bar{4}$, $\bar{3}^{-1}=\bar{5}$, $\bar{4}^{-1}=\bar{2}$, $\bar{5}^{-1}=\bar{3}$, $\bar{6}^{-1}=\bar{6}$.

*3 □□□□□□□ F□□□□.□□□□□□□□□□□□□□□□□□□□□.□

$$\sigma:\ F \to C$$
$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \longmapsto a+bi,$$

□□ σ□□□,□□□ σ□□□□□□□,□□ σ□ F□ C□□□□□□.

4 □□□□□□□ F□□□□,□□ $\mathrm{char}\,F=3$.

5 □□□□□□.

6 $Z_2[x]$□□□□□□□:x, $x+1$;□□□□□□□□□□ x^2+x+1(□□ 0□ 1□□□□□□□□□,□□□□□□□□□.□□□□ Z_2□□□□).□□□□□ $\bar{0}, \bar{1}$□□□□ 0, 1.□□□□□□.

7. □ $\bar{f}(x)$□□ $f(x)$□□□□□□ 2□□□□□ $Z_2[x]$□□□□□:

$$\bar{f}(x)=x^5+x^2+1=x^2(x+1)(x^2+x+1)+1,$$

□□□□□□$\bar{f}(x)$□ $Z_2[x]$□□□□.□□□□□□□□□ $f(x)$□ Q□□□□.

8 □ $f(x)$□□□□□□ 2□□ $Z_2[x]$□□□□□:

$$\bar{f}(x)=x^4+x^3+x^2+1=(x+1)(x^3+x+1).$$

□□□□ x^3+x+1□ Z_2□□□□.□□□□□$\bar{f}(x)$□ $Z_2[x]$□□□□□□□□□.□□ $f(x)$□ Q□□□,□

$$f(x)=f_1(x)f_2(x),\quad \deg f_i(x)<\deg f(x),\quad i=1,2.$$

□□□□□□□□□□□□□□□□ 2□□,□

$$\bar{f}(x)=\bar{f}_1(x)\bar{f}_2(x),\quad \deg\bar{f}_i(x)<\deg\bar{f}(x),\quad i=1,2.$$

□$\bar{f}(x)$□ $Z_2[x]$□□□□□□□□□□,$\bar{f}_1(x)$□$\bar{f}_2(x)$□□□□□□□□□□.□□ $f_1(x)$□ $f_2(x)$□□□□□□□□□□.□□□□ $f(x)$□□□□.□□ $f(x)$□□□□□□□□ $\pm1,\pm5$.□□□□□□□□□ $f(x)$□□.□□□□□□ $f(x)$□ Q□□□□.

□ 8□ □□□□

□ □ 8 1

1 (1) □□.□□: $2x^2+(-2x^2+x)=x$,□□ 2□□□□.

(2) □. (3) □.

2 (1) □□□□.□□:□□□□□□□ $\cos 2x=2\cos^2 x-1$.

(2) □□□□.□□:□ $k_0+k_1\cos x+k_2\cos 2x+k_3\cos 3x=0$,□□ x□ 4□□□□□,

□□□□□□ k_0,k_1,k_2,k_3□ 4□□□□□□□□□□□,□□□□□□□.

(3) □□□□.□□:□□□□(2)□□□□□.

(4) □□□□.□□:□□□.□□□□(2)□□□□;

□□□.□ $k_1\sin x+k_2\cos x+k_3\sin^2 x+k_4\cos^2 x=0$,

□□□□□□□ 1□□2□□3□□□,□□□ x□□□□□□□,□□□□ 4□□□,□□ k_1,k_2,k_3,k_4□ 4□□□□□□□□□□□,□□□□□□□.

(5) □□□□. (6) □□□□.

3 □□□□□□□ a□1,□ a□□□□□□□□□□□,□□□□□ 1.

4 $1,i$□□□□,□□□ 2.□□ $z=a+bi$□□ $1,i$□□□□□(a,b).

5 1□□□□,□□□□□ 1.

6 $E_{11},E_{22},\ldots,E_{nn},E_{12}+E_{21},\ldots,E_{1n}+E_{n1},\ldots,E_{n-1,n}+E_{n,n-1}$□ V_1□□□□,□□ V_1□□□□$\dfrac{n(n+1)}{2}$.

7. V_2□□□□□:

$$E_{12}-E_{21},\ldots,E_{1n}-E_{n1},E_{23}-E_{32},\ldots,E_{2n}-E_{n2},\ldots,E_{n-1,n}-E_{n,n-1}.$$

V_1□□□□$\dfrac{n(n-1)}{2}$.

8 W□□□□□:

$$E_{11},\ E_{12},\ \dots,\ E_{1n},\ E_{22},\ E_{23},\ \dots,\ E_{2n},\ \dots,\ E_{n-1,n-1},\ E_{n-1,n},\ E_{nn}.$$

W□□□□ $\dfrac{n(n+1)}{2}$.

9 □□:□ $A=(\alpha_1,\alpha_2,\alpha_3)$, $B=(\beta_1,\beta_2,\beta_3)$, □□

$$(\beta_1,\beta_2,\beta_3)=(\alpha_1,\alpha_2,\alpha_3)P,$$

□□ $B=AP$. □□□□□□□□□□

$$P=\begin{pmatrix}0&1&1\\-1&-3&-2\\2&4&4\end{pmatrix}.$$

□□ $\alpha=(\beta_1,\beta_2,\beta_3)Y$, □□ $BY=\alpha$. □□, □ $Y=(1,0,2)$□. □□

$$X=PY=(2,-5,10)□.$$

10 □□:□ V□□□□□$\delta_1,\delta_2,\dots,\delta_n$, □□□□□□, $\delta_1,\delta_2,\dots,\delta_n$ □□ $\alpha_1,\alpha_2,\dots,\alpha_n$ □□□□, □□ $\mathrm{rank}\{\delta_1,\delta_2,\dots,\delta_n\}$□$\mathrm{rank}\{\alpha_1,\alpha_2,\dots,\alpha_n\}$, □□□□ $\mathrm{rank}\{\alpha_1,\alpha_2,\dots,\alpha_n\}$, □□□□ $\alpha_1,\alpha_2,\dots,\alpha_n$ □□□□□.

11 □□ f□F^X, f□□□□□□□□ $f(x_1), f(x_2),\dots,f(x_n)$□□. □□ n□□□□□:

$$f_i(x_j)=\delta_{ij},\ 1□j□n,\ i=1,2,\dots,n.$$

□□ f□□□ $f_1,f_2,\dots,f_n$□□□□;□□□ $f_1,f_2,\dots,f_n$□□□□. □□ $f_1,f_2,\dots,f_n$□F^X□□□□, □□ $\dim F^X=n$. □□ f□□□□□□□□□$(f(x_1),f(x_2),\dots,f(x_n))$□.

12 (1) □□□□□□□□□□□□□;

(2) V□□□□□

$$\begin{pmatrix}1&0\\0&-1\end{pmatrix},\ \begin{pmatrix}0&1\\1&0\end{pmatrix},\ \begin{pmatrix}0&i\\-i&0\end{pmatrix},\quad \dim V=3;$$

(3) □□□□□□□□□

$$(x_1,x_2,x_3)□.$$

□ □ 8 2

1 (1) □; (2) □□; (3) □□.

2 □□:□□ $C(A)$□□□,□□□□□,□□□□□.

3 □□:□□□□ 4 2□□ 1□□□□. $C(A)$□□□□□ $E_{11}, E_{22},\dots,E_{nn}$; □□ $\dim C(A)=n$.

4 □ $X=(x_{ij})$□$C(A)$, □ $AX=XA$. □□□□□□ 9□□□□□□□□. □□□□ x_{11},x_{12},x_{13}□□□□□□,□□□□□ $C(A)$□□□□□

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\ \begin{pmatrix}0&2&0\\1&4&-1\\2&0&0\end{pmatrix},\ \begin{pmatrix}0&0&1\\1&0&0\\4&-2&1\end{pmatrix}.$$

5 □□:□□ α□□β,γ□, β□□α,γ□.

6 □□□ 3,□□□□ $\alpha_1,\alpha_2,\alpha_3$.

7. □□:□ A□□□□□ $A_1, A_2, \dots, A_s$ □□□□□□□□□□□ $A_{j_1}, \dots, A_{j_r}$,□□ $\beta_{j_1},\dots,\beta_{j_r}$□□□□$\beta_1,\beta_2,\dots,\beta_s$□□□□□□□□□.

*8 □□:□□□□□ $s>0$,□□ $p>(n-p)$.□□

$$\alpha_1=(1,0,\dots,0,1,0,0,\dots,0,0),$$

□ $p+1$ □

$$\alpha_2=(0,1,\dots,0,0,1,0,\dots,0,0),$$

…………

$$\alpha_{n-p}=(0,0,\dots,0,1,\dots,0,0,0,0,\dots,0,1)$$

□□ $f(x_1,x_2,\dots,x_n)=0$□□,□□□□□□□□□□□ $k_1\alpha_1+k_2\alpha_2+\dots+k_{n-p}\alpha_{n-p}$□□ $f(x_1,x_2,\dots,x_n)=0$□□.□□□$\alpha_1,\alpha_2,\dots,\alpha_{n-p}$□□ W.

9 □□:□□ V_1□ V,□□□□ $\alpha|$ V_1.□□ $\alpha|$ V_2,□ $\alpha|$ V_1□ V_2.□□ α□ V_2,□□□ $\beta|$ V_2.□ $\beta|$ V_1,□ $\beta|$ V_1□ V_2.□ β□ V_1,□□ $\alpha+\beta|$ V_1□ V_2.

*10 □□:□ s□□□□□□.□ $s=2$□,□ 9□□□□ V_1□ V_2□ V.□□□□□□ $s-1$ □□□□□,□□ s□□□.□□□□□,V□□□□□ $\alpha|$ V_1□ V_2□…□ V_{s-1}.□□ $\alpha|$ V_s,□ α $|$ V_1□ V_2□…□ V_{s-1}□ V_s.□□ α□ V_s,□ $\beta|$ V_s.□□□□ $W=\{k\alpha+\beta\,|\,k$□ $F\}$.□□□,□□□ k□ F,□ $k\alpha+\beta|$ V_s.□□□□□□□ $V_i(i=1,2,\dots,s-1)$, W□□□□□□□□ V_i.□□ W□□□□□□□ γ□□□ V_1□ V_2□…□ V_{s-1}.□□ $\gamma|$ V_1□ V_2□…□ V_{s-1}□ V_s.

11 V_1+V_2□□□□□ $\alpha_1,\alpha_2,\beta_1$, $\dim(V_1+V_2)=3$;

V_1□ V_2□□□□□ $(0,1,1,-1)$□, $\dim(V_1$□ $V_2)=1$.

12 V_1+V_2□□□□□ $\alpha_1,\alpha_2,\beta_1$, $\dim(V_1+V_2)=3$;

V_1□ V_2□□□□□ $(5,-1,5,2)$□, $\dim(V_1$□ $V_2)=1$.

13 □□:□□ $V=V_1+V_2$,□□□□ V□□□□□ $\alpha=(a_1,a_2,\dots,a_n)$□□□□□ $\alpha_1+\alpha_2$,□□ α_1□ V_1,α_2□ V_2.□□

$$\alpha_2=\left(\frac{1}{n}\sum_{i=1}^{n}a_i,\frac{1}{n}\sum_{i=1}^{n}a_i,\dots,\frac{1}{n}\sum_{i=1}^{n}a_i\right).$$

□□□ V_1□ $V_2=0$.

14 □□:□ V□□□□$\alpha_1,\alpha_2,\dots,\alpha_n$.□□□

$$V=□\alpha_1□+□\alpha_2□+\dots+□\alpha_n□,$$

□□□□□□□□□□□□.

15 (1) □□:□□ $M_n^0(K)$□□□,□□□□□,□□□□□;

(2) □□:□□ $M_n(K)=□I□+M_n^0(K)$,□□□□□□ n□□□ $A=(a_{ij})$□□□□ A_1+A_2,□□ A_1□□I□,A_2□ $M_n^0(K)$.□□□I□□ $M_n^0(K)=0$.

*16 □□:□□□.□□□□□§ 6□□□ 5.

□□□.□ A□□□□,□□□□§ 6□□□ 5□,

$$\dim V_{\lambda_1}+\dim V_{\lambda_2}+\dots+\dim V_{\lambda_s}=n.$$

□ $V_{\lambda_1},V_{\lambda_2},\dots,V_{\lambda_s}$ □□□□□，□□□□□ n □□□□□□□□，□□ K^n □□□□.
□□ $K^n=V_{\lambda_1}+V_{\lambda_2}+\dots+V_{\lambda_s}$. □□ $V_{\lambda_1}+V_{\lambda_2}+\dots+V_{\lambda_s}$ □□□.

17. (1) □□：□□□ $W_1\subseteq W$, $W_2\subseteq W$.

(2) □□：□□ $(f_1(x),f_2(x))=1$, □□□ $u_i(x)$ □ $K[x]$ $(i=1,2)$，□□ $u_1(x)f_1(x)+u_2(x)f_2(x)=1$. □□□ x □ A □□□，$u_1(A)f_1(A)+u_2(A)f_2(A)=I$，□□□□□□□ $W\subseteq W_1+W_2$，□□ W_1 □ $W_2=0$.

□□ 8 3

1. □□：□□□□□□ x，□ $\sigma: x\longmapsto 2^x$，□□ σ □ R □ R^+ □□□□□□□.

2. □□：□□ $\dim M_{s\times n}(F)=sn=\dim F^{sn}$，□□ $M_{s\times n}(F)\cong F^{sn}$，□□□□□□ σ □：□ $A=(a_{ij})$，$\sigma(A)=(a_{11},a_{12},\dots,a_{1n},a_{21},a_{22},\dots,a_{2n},\dots,a_{s1},a_{s2},\dots,a_{sn})$.

3. □□：□□ $\dim F[x]_n=n=\dim F^n$，□□ $F[x]_n\cong F^n$. □□□□□□ σ □：□□ $f(x)=a_0+a_1x+\dots+a_{n-1}x^{n-1}$，$\sigma(f(x))=(a_0,a_1,\dots,a_{n-1})$.

4. □□：□ $U=$□$\alpha_1,\alpha_2,\alpha_3,\alpha_4$□，□ $\dim U=4$，□ σ □ U □ F^4 □□□□□□□，□□ U □□□□□□□□□□□□□□□ $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ □□□□. □□ $\sigma(\alpha_1+\alpha_2),\sigma(\alpha_2+\alpha_3)$, $\sigma(\alpha_3+\alpha_4),\sigma(\alpha_4+\alpha_1)$，□□□□□□□□□□□□□□□□□□□□□□□□，□□□□ $\sigma(\alpha_1+\alpha_2),\sigma(\alpha_2+\alpha_3),\sigma(\alpha_3+\alpha_4),\sigma(\alpha_4+\alpha_1)$ □□□□□□，□□□□□□□□□□□ F^4 □□□□□□□□□□□□，□□□□□ W □□□□□□：□□ $\alpha_1+\alpha_2,\alpha_2+\alpha_3,\alpha_3+\alpha_4$，□□□ 3.

5. (1) □□：□□ L □□□，□□□ L □□□□□□□□□□□□□. L □□□□□ $E_{11}+E_{22}$, $E_{12}-E_{21}$; $\dim L=2$.

(2) □□：□□ $\dim_R C=2=\dim L$，□□ $C\cong L$. □□□□□□ σ □：

$$\sigma(a+bi)=\begin{pmatrix}a&b\\-b&a\end{pmatrix}.$$

*6. (1) □□：□□□□□□□□□□□□□；

(2) □□□□ $\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$, $\begin{pmatrix}i&0\\0&-i\end{pmatrix}$, $\begin{pmatrix}0&i\\i&0\end{pmatrix}$, $\dim H=4$;

(3) □□：□□ $\dim H=4=\dim R^4$；□□□□□ τ □：□ $\alpha=a+bi$, $\beta=c+di$, □

$$\tau\begin{pmatrix}\alpha&\beta\\-\bar{\beta}&\bar{\alpha}\end{pmatrix}=(a,c,b,d).$$

□□ 8 4

1. □□：□□ $w\longmapsto w+U$ □ W □ V/U □□□□□□□.

2. □□：V/U □□□□ U □□□□□□□□□□□. □ U □□□□□ u_1，□□□□□ V

□□□□: u_1, α_1, α_2, □□ V/U □□□□□ $\alpha + U$ □□□ $\alpha_1 + U, \alpha_2 + U$ □□□□, □□ $\alpha_1 + U, \alpha_2 + U$ □□□□. □□ $\alpha_1 + U, \alpha_2 + U$ □ V/U □□□□, □□ $\dim V/U = 2$. □□□□□□□:

$$\dim V/U = \dim V - \dim U = 3 - 1 = 2.$$

3 □□: □□ Ω_2 □□□□□ W/U, □ $\dim W = \dim U + \dim W/U = n - 1$. □□ W □ V □ $n-1$ □□□□, □ W □□ U. □□ W □ Ω_1. □□□□ σ □□□. □□ Ω_2 □, $W_1/U = W_2/U$, □□□ W_1/U, W_2/U □□□□□: $\alpha_1 + U, \ldots, \alpha_{n-3} + U; \beta_1 + U, \ldots, \beta_{n-3} + U$. □ U □□□□□ u_1, u_2, □□□□,

$$W_1 = □\alpha_1, \ldots, \alpha_{n-3}, u_1, u_2□, \quad W_2 = □\beta_1, \ldots, \beta_{n-3}, u_1, u_2□.$$

□□ $W_1/U = W_2/U$, □□ $\alpha_i + U$ □ W_2/U, $i = 1, 2, \ldots, n-3$. □□□□□ $□\alpha_1, \ldots, \alpha_{n-3}, u_1, u_2□ = □\beta_1, \ldots, \beta_{n-3}, u_1, u_2□$. □ $W_1 = W_2$. □□ σ □□□.

4 □□: □

$$\sigma: \quad U + W/W \quad U/U \cap W$$

$$(u + w) + W \longmapsto u + U \cap W,$$

□□ σ □□□□□□□□□, □□□□□□□□□□□.

□ 9 □ □□□□

□ □ 9 1

1 (1) □□. (2) □. 2 (1) □; (2) □. 3 □. 4 □.

*5 □(□□: □□□□□□□□□□□□□).

*6 □(□□: □□□□□□ 2 □□□, □ $(x + y)^2 = x^2 + y^2$).

7. (1) □□: □□□□□□□□□□□. (2) □□: □□□□.

8 □□: □□□. □□ V □□□□□□ A □□□□□□ A □ V □□□□□□□□□. □□□. □□ A □□□, □□.

9 □□: □ $k_0\alpha + k_1 A\alpha + k_2 A^2\alpha + \ldots + k_{m-1} A^{m-1}\alpha = 0$, □□□□□ A^{m-1} □□, □ $k_0 A^{m-1}\alpha = 0$, □□□□ $k_0 = 0$. □□□ $k_1 A\alpha + k_2 A^2\alpha + \ldots + k_{m-1} A^{m-1}\alpha = 0$.

10 □□: □ k □□□□□□.

11 (1) □□: □□□. □ $(A + B)^2 = A + B$ □□□, $AB + BA = 0$, □□□□□□ A, □□ $AB = -BA$ □□, □□□ $AB = BA$.

(2) □□: □□ $(A + B - AB)^2 = A + B - AB$.

□ □ 9 2

1 □□: □ $\mathrm{Ker}\,A, \mathrm{Im}A$ □□□□□□□.

2 □□: □□ $V = \mathrm{Ker}\,A + \mathrm{Im}A$, □□□□ α □ V, □□ $\alpha - A\alpha$ □ $\mathrm{Ker}\,A$. □□ $\mathrm{Ker}\,A$ □ $\mathrm{Im}A$

$=0$.

3 □□：□□ B□ AV□□□□：$B|AV$，□□□□ $B(AV)=(BA)V$．□□ $\dim Ker(BA)=\dim V-\dim(BA)V=\cdots$.

4 □□：□□□□□□ 3□□ 3□□□□.

5 (1) □□：□□□．□□□ 2□□□□□□ §1□□□ 2.□□□．□□ δ□ $Im\, A$，□□□ α□ V□□ $A\alpha=\delta$．□□ $BA=A$□□□ δ□ $Im\, B$.

(2) □□：□□□．□□□ 2□□□□□，□□ α□ V，□ $\alpha=\alpha_1+\alpha_2$，□□ α_1□ $Ker\, B$，α_2□ $Im\, B$，□□ $B\alpha=B\alpha_2=\alpha_2$．□□ $B\alpha=\alpha-\alpha_1$，□□□□□□ A，□□□ $AB=A$．□□□．□□ δ□ $Ker\, A$，□□ $BA=B$□□ δ□ $Ker\, B$.

6 □□：□ $U=Ker\, A$，□□ V□□□□□，□□□□ U□□□□ W．□ $\dim V=n$，$\dim W=r$．□□ $\dim AV=r$．□□ $W\cong AV$．□ $M=AV$.

*7. □□：□ i□j□，□□ A_i□ A_j，□□ A_i-A_j□0，□□ $Ker(A_i-A_j)$□ V．□□□□□□ 8 2□□ 10□□□□.

□ □ 9 3

1 $\begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.　2 $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.　3 $\begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}$.

4 $\begin{pmatrix} a & c & 0 & 0 \\ b & d & 0 & 0 \\ 0 & 0 & a & c \\ 0 & 0 & b & d \end{pmatrix}$.

5 □□：□□□ 9 1□□ 9□□□□，$\alpha, A\alpha, \dots, A^{n-1}\alpha$ □□□□，□□ A□□ $A^{n-1}\alpha, A^{n-2}\alpha, \dots, A\alpha, \alpha$□□□□.

6 □□：□□□□□□□□□□□□□□□□□□□□□□□□□□□.

7. (1) □□：□□ $\dim Hom(V,V)=(\dim V)^2=n^2$，□□ V□□ n^2+1□□□□□□□□□□□，□□□ F□□□□ 0□□□ $k_0, k_1, \dots, k_{n^2}$，□□

$$k_0 I+k_1 A+k_2 A^2+\dots+k_{n^2} A^{n^2}=0.$$

□ $f(x)=k_0+k_1x+k_2x^2+\dots+k_{n^2}x^{n^2}$，□□□ x□ A□□.

(2) □□：□□ $u(x), v(x)$□ $F[x]$，□□

$$u(x)f(x)+v(x)g(x)=d(x),$$

□□□ x□ A□□.

(3) □□：□□□．□ $f(x)=a_0+a_1x+\dots+a_mx^m(a_0$□$0)$□□ $f(A)=0$．□□□ x□ A□□，□□□□□□□□□□□□□□□．□□□．□□(1)□□，□□□□□□□□□□□□□□□ $f(x)$□□ $f(A)=0$，□□ $f(x)$□□□□□□□□ 0.

8 □□：□□□□□□，

$$(A\alpha_1, A\alpha_2, \dots, A\alpha_n) = (\alpha_1, \alpha_2, \dots, \alpha_n) A.$$

□□□ 8□§3□1□□□;□□□ rank A □□□.

*9 □□:□ $\dim V = n$,□ Ker $\mathcal{A}$□ V□□□□□ W,□□ $V = \mathrm{Ker}\,\mathcal{A} \oplus W$.□ W□□□□□ $\alpha_1, \dots, \alpha_r$,□ Ker $\mathcal{A}$□□□□□:$\alpha_{r+1}, \dots, \alpha_n$,□□□□ V□□□□.□□□□□□ 9.2□□ 6□□□□□□□□,V□$= \mathcal{A}V \oplus N$.□□ $\mathcal{A}\alpha_1, \dots, \mathcal{A}\alpha_r$□ $\mathcal{A}V$□□□□,□□ N□□□□□,□□□□ V□□□□□.□□ $\mathcal{A}$□ V□□□□□ V□□□□□□□□□.

10 □□:□□□ $\varepsilon_1, \varepsilon_2, \varepsilon_3$ □□ η_1, η_2, η_3 □□□□□ S,□□□ S^{-1},□□□ $B = S^{-1}AS$,□□

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

11 □□:□ $C = (\alpha_1, \alpha_2, \alpha_3)$, $T = (\eta_1, \eta_2, \eta_3)$.□ $(\eta_1, \eta_2, \eta_3) = (\alpha_1, \alpha_2, \alpha_3) S$,□ $S = C^{-1}T$.□□ $B = S^{-1}AS = (C^{-1}T)^{-1}A(C^{-1}T)$,□□□

$$B = \begin{pmatrix} 1 & 2 & 2 \\ 3 & -1 & -2 \\ 2 & 3 & 1 \end{pmatrix}.$$

12 (1) □□:$(\alpha_2, \alpha_3, \alpha_1) = (\alpha_1, \alpha_2, \alpha_3) \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$,

□□□□ S□□□□□,□□ $S^{-1} = S$□.□□ $S^{-1}AS$,□

$$\begin{pmatrix} a_{22} & a_{23} & a_{21} \\ a_{32} & a_{33} & a_{31} \\ a_{12} & a_{13} & a_{11} \end{pmatrix};$$

(2) □□:

$$(k\alpha_1, \alpha_2, \alpha_3) = (\alpha_1, \alpha_2, \alpha_3) \begin{pmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

□□□□ S□□□□□,□□ $S^{-1}AS$,□

$$\begin{pmatrix} a_{11} & k^{-1}a_{12} & k^{-1}a_{13} \\ ka_{21} & a_{22} & a_{23} \\ ka_{31} & a_{32} & a_{33} \end{pmatrix};$$

(3) □□:$(\alpha_1, \alpha_1 + \alpha_2, \alpha_3) = (\alpha_1, \alpha_2, \alpha_3) \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

□□ $S^{-1}AS$,□

$$\begin{pmatrix} a_{11} - a_{21} & a_{11} + a_{12} - a_{21} - a_{22} & a_{13} - a_{23} \\ a_{21} & a_{21} + a_{22} & a_{23} \\ a_{31} & a_{31} + a_{32} & a_{33} \end{pmatrix}.$$

* 13 □□:□ A□ V□□□□□$\alpha_1,\alpha_2,\ldots,\alpha_n$□□□□□ A.□□□ F□□□ n□□□□□ S,□

$$(\eta_1,\eta_2,\ldots,\eta_n)=(\alpha_1,\alpha_2,\ldots,\alpha_n)S.$$

□ $\eta_1,\eta_2,\ldots,\eta_n$ □□ V□□□□,□ A□□$\eta_1,\eta_2,\ldots,\eta_n$□□□□ $B=S^{-1}AS$.□□□□□□,$S^{-1}AS=B=A$.□□ $AS=SA$.□ $S=\mathrm{diag}\{1,2,\ldots,n\}$,□ $AS=SA$,□□□□□ 4□□□ 4 2□□ 1□□□□,A□□□□□□□.□ $S=I+E_{1j}$(□□ 2□ j□ n),□ $AS=SA$□□□ A□□□□□□.

14 (1)

$$\begin{array}{cccc} 2 & -3 & 3 & 2 \\ -\frac{2}{3} & -\frac{4}{3} & \frac{10}{3} & \frac{10}{3} \\ \frac{8}{3} & -\frac{16}{3} & \frac{40}{3} & \frac{40}{3} \\ 0 & 1 & -7 & -8 \end{array}.$$

(2) □□:□ α □□$\alpha_1,\alpha_2,\alpha_3,\alpha_4$□□□□□ X,□

$$\alpha\in \mathrm{Ker}\,A \Leftrightarrow A\alpha=0 \Leftrightarrow AX=0$$

□□□□□□□□□ $AZ=0$□□□□□□□,□□□□□□□□□□□□ Ker A□□□□.

□ β □□$\alpha_1,\alpha_2,\alpha_3,\alpha_4$□□□□□ Y,□

$$\beta\in AV \Leftrightarrow \beta\in\langle A\alpha_1, A\alpha_2, A\alpha_3, A\alpha_4\rangle$$

$$\Leftrightarrow Y\in\langle A_1, A_2, A_3, A_4\rangle$$

□□ A□□□□□A_1, A_2, A_3, A_4□□□□□,□□□□□□□□□□□□ AV□□□□.

Ker A□□□□□:$4\alpha_1+3\alpha_2-2\alpha_3,\alpha_1+2\alpha_2-\alpha_4$;

AV□□□□□:$\alpha_1-\alpha_2+\alpha_3+2\alpha_4,2\alpha_2+2\alpha_3-2\alpha_4$.

(3) □□:Ker A□□□□□□□□(2)□□□□□□,□□□□□□□□□□□ 1□2□,□□□ 3□4□,□ 4□□□□□□□□□ 0.□□ 3□4□□□□□□□□□ Ker A□□□□□,□□ V□□□□.

(4) □□:□□□(2)□□□□□□ ImA□□□□,□□□□□□(3)□□□□□,□□□□ V□□□□.

□ □ 9 4

1 (1) A□□□□□□□□ 1(□□),10.

A□□□□□□ 1□□□□□□□□□

$$\{k_1(-2\alpha_1+\alpha_2)+k_2(2\alpha_1+\alpha_3)\mid k_1,k_2\in K,□\ k_1,k_2□□□0\};$$

A□□□□□□ 10□□□□□□□□□

$$\{k(\alpha_1+2\alpha_2-2\alpha_3)\mid k\in K,□\ k□0\};$$

(2) A□□□□□□□ 1,3(□□).

A□□□ 1□□□□□□□□□

$$\{k(-2\alpha_1+\alpha_3)\mid k\in K,□\ k□0\};$$

A□□□ 3□□□□□□□□□

$\{k(\alpha_1-\alpha_2+2\alpha_3)\mid k$□ $K,$□ k□ $0\}.$

2 (1) A□□□□, A□□□□□

$$\begin{matrix}1&0&0\\0&1&0\\0&0&10\end{matrix};$$

(2) A□□□□□□.

3 (1) A□□□□□□□ 1(□□), 0(□□).

A□□□ 1□□□□□□□□

$\{k_1(\alpha_1+\alpha_3)+k_2\alpha_4\mid k_1, k_2$□ $K,$□ k_1, k_2□□□ $0\};$

A□□□ 0□□□□□□□□

$\{l_1\alpha_2+l_2\alpha_3\mid l_1, l_2$□ $K,$□ l_1, l_2□□□ $0\}.$

(2) A□ V□□□□ $\alpha_1+\alpha_3, \alpha_4, \alpha_2, \alpha_3$□□□□□

$$\begin{matrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{matrix}.$$

4 □□:□ λ_1, λ_2□ A□□□□□□, ξ_1, ξ_2□□□ A□□□ λ_1, λ_2□□□□□□□. □□ $k_1\xi_1+k_2\xi_2=0$, □□ $k_1=k_2=0$.

5 □□:□□□□, □□□□□ 4□□□□.

6 □□:□□ A□□□□□□□□□ λ_1, λ_2, □□□ 5□□□□, □□□□□□□□.

7. (1) □□:□□□□, □□ A□□□□□.

(2) □□:□□ $A\xi=\lambda\xi$□□□□□□ A^{-1}□□□.

8 □□: A□□□□□□□□ V□□□□ A□□□□□□□□□□□□□□: $\xi_{i1}, \xi_{i2}, \ldots, \xi_{ir_i}$, $i=1, 2, \ldots, s$; □□ $\xi_{i1}, \xi_{i2}, \ldots, \xi_{ir_i}$□ A□□□ λ_i□□□□□□□□□□□.

*9 □□:□□ A□□□□, □□ V□□□□□□ $\alpha_1, \alpha_2, \ldots, \alpha_n$, □□ A□□□□□□□ $A=\mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. □□ B□ A□□□, □□ B□□□□□□□ B□□□□□□. □ $B=c_0I+c_1A+\cdots+c_{n-1}A^{n-1}$, □ $B=c_0I+c_1A+\cdots+c_{n-1}A^{n-1}$. □□ $(c_0, c_1, \ldots, c_{n-1})$□□□□□.

□ □ 9 5

1 □□:□□ $A(\alpha_1+2\alpha_2)$, $A(\alpha_2+\alpha_3+2\alpha_4)$.

2 (1) □□:□□□ $A|W$□□□, □□□□ $A|W$□□□□□□□□□ $A|W$□□□.

(2) □□:□□(1)□□□□□, □□□□□□ W□ A^{-1}□□□□□□. □□ β□ W, □ $(A|W)^{-1}\beta=\gamma$, □ γ□ W. □□ $\gamma=(A^{-1}|W)\beta$.

3 □□:□ A□□□□□□ λ_0, □ V_{λ_0}□ B-□□□. □ $B|V_{\lambda_0}$□□□□□□ μ_0, □□□ ξ□ V_{λ_0}, □□ $B\xi=(B|V_{\lambda_0})\xi=\mu_0\xi$.

4 (1) □□:□□ α_n □ W,□□□ W □ A-□□□,□□ $A\alpha_n$ □ W,□ $\alpha_{n-1}+a\alpha_n$ □ W,□□ α_{n-1} □ W.□□□ $A\alpha_{n-1}$ □ W,□□□□.

(2) □□:□ W □ A □□□□□□□□□,□ β □ W,□ β □ 0.□ $\beta=k_1\alpha_1+\dots+k_s\alpha_s$,□□ k_s □ 0.□ $A\beta$ □ W □□□ $k_2\alpha_1+\dots+k_s\alpha_{s-1}$ □ W.□□□ A □□□□.

(3) □□:□□□(2)□□□□□.

(4) □□:□ W □□□□ A-□□□,□ α_1 □ W.□ $\dim W=m$,□ W □□□□□ $\alpha_1,\beta_2,\dots,\beta_m$.□

$$\beta_2=k_{21}\alpha_1+k_{22}\alpha_2+\dots+k_{2s}\alpha_s,$$

$$\dots\dots\dots\dots$$

$$\beta_m=k_{m1}\alpha_1+k_{m2}\alpha_2+\dots+k_{ms}\alpha_s,$$

□□ $k_{2s},\dots,k_{ms}$ □□□□,□□□ k_{2s} □ 0.□□ $\alpha_1,\beta_2,\dots,\beta_m$ □□ $\alpha_1,\alpha_2,\dots,\alpha_s$ □□□□,□□ m □ s.□□ $A\beta_2$ □ W,□□ $k_{21}(a\alpha_1)+k_{22}(\alpha_1+a\alpha_2)+\dots+k_{2s}(\alpha_{s-1}+a\alpha_s)$ □ W.□□□□,$k_{23}\alpha_2+\dots+k_{2s}\alpha_{s-1}$ □ W.□□□ A □□□□,□□□ $k_{2s}\alpha_2$ □ W,□□ α_2 □ W.□□□□ α_3 □ $W,\dots,\alpha_s$ □ W.□□ $\alpha_1,\alpha_2,\dots,\alpha_s$ □□ $\alpha_1,\beta_2,\dots,\beta_m$ □□□□,□□ s □ m □□ $s=m$ □□ $\alpha_1,\alpha_2,\dots,\alpha_m$ □ W □□□□.□,$W=$□$\alpha_1,\alpha_2,\dots,\alpha_m$□.□□ A □□□□□□□□□□ $n+1$ □,□□□:

$$\{0\},□\alpha_1□,□\alpha_1,\alpha_2□,\dots,□\alpha_1,\alpha_2,\dots,\alpha_n□.$$

5 □□:□ A □□□□,□ A □□□□□□ λ_0,□ ξ □ A □□□ λ_0 □□□□□□□,□□□□ ξ □□ A □ 1 □□□□□□.□ A □□□□□,□ A □ V □□□□ $\alpha_1,\alpha_2,\dots,\alpha_n$ □□□□□ A.□ A □□□□□□□□□,□ A □□□□.□□ A □□□□□□□□□□□□□□,□□□□□ A □□□□□□□□□□□.□ $z,\bar{z}$ □ A □□□□□□,□□ $z=a+bi$, a,b □ R.□ $X=X_1+iX_2$ □ A □□□ z □□□□□□□,□ $\overline{X}=X_1-iX_2$ □ A □□□ $\bar{z}$ □□□□□□□,□□ X_1,X_2 □ R^n.□ $AX=zX$, $A\overline{X}=\bar{z}\overline{X}$ □□□,

$$AX_1=aX_1-bX_2,\quad AX_2=bX_1+aX_2.$$

□ $\beta_1=(\alpha_1,\alpha_2,\dots,\alpha_n)X_1,\beta_2=(\alpha_1,\alpha_2,\dots,\alpha_n)X_2$.□□□ β_1,β_2 □□ A □□□□□□;□□□ β_1,β_2 □□□□.

6 □□:□□□□,□□ A □□□□□□.

*7. □□:V_{λ_0} □ A-□□□,□ V_{λ_0} □ 0.□ A □□□□□,□□□□□□□.□□□ A □□□□□□,□□ V_{λ_0} □ V.□ V_{λ_0} □□□□ $\alpha_1,\dots,\alpha_r$,□□□□□ V □□□□ $\alpha_1,\alpha_2,\dots,\alpha_n$,□ A □□□□□□□ $A=\begin{matrix} A_1 & A_3 \\ 0 & A_2 \end{matrix}$,□□ A_1 □ $A|V_{\lambda_0}$ □ V_{λ_0} □□□□ $\alpha_1,\dots,\alpha_r$ □□□□,□□ $A_1=\lambda_0 I_r$.□□ $|\lambda I-A|$.

*8 □□:□□ V □□□□□ n □□□□,□□ A □□□□□ $f(\lambda)=|\lambda I-A|=(\lambda-\lambda_1)^{l_1}\dots(\lambda-\lambda_s)^{l_s}$,□□ $\lambda_1,\dots,\lambda_s$ □ A □□□□□□□□.□□,$l_1+\dots+l_s=n$.□ $\dim V_{\lambda_i}=r_i$, $i=1,2,\dots,s$.□□□ 7 □□□□□,r_i □ l_i, $i=1,2,\dots,s$.□□□□ A □□□□□□□□□□□□□□(□□□□□□□□□□□□).

9 □□: A□□□□□ $\Leftrightarrow$ $A^2 = A$ $\Leftrightarrow$ $A(A-I) = 0$ $\Leftrightarrow$ $\mathrm{Ker}[A(A-I)] = \mathrm{Ker}\,0 = V$. □□$(x, x-1) = 1$, □□□□□□ 7□□□.

*10 □□: □ V□□ F□ n□□□□□, V□□□□□ $\alpha_1, \alpha_2, \ldots, \alpha_n$. □□ V□□□□□□□□ A, □□ $A(\alpha_1, \alpha_2, \ldots, \alpha_n) = (\alpha_1, \alpha_2, \ldots, \alpha_n) A$, □ $A^2 = I$ $\Leftrightarrow$ $A^2 = I$. □□□□□□ 9□□□□.

11 $f(x) = x - k$.

12 □□:□□□□□□□□□□□□□□□.

13 □□: $A^2 = 0$.

14 □□: $A^n = 0$.

*15 □□:□□ $\bar{A}$□ V/W□ V/W□□□□□(□□□□□□□ $\bar{A}$□□□ $\alpha + W$□□□□□□□□), □□□□ $\bar{A}$□□□□□□□□□□□□□□.

□ □ 9 6

1 □□:□□§5□□□ 8□ Hamilton-Cayley□□.

2 □□:□□ Hamilton-Cayley□□, □□ A□□□□□□□□□□□□□$(-1)^n|A|$.

3 (1) $\lambda^2 - 5\lambda + 6$; (2) $\lambda^2 - 2\lambda + 2$.

*4 □□:□□□.□□□□.□□ A□ B□□□□□□λ_0, □□□ $\alpha\in C^n, \beta\in C^m$ □ $\alpha\neq 0, \beta\neq 0$, □□ $A\alpha = \lambda_0\alpha$, $B'\beta = \lambda_0\beta$. □ $C = \alpha\beta'$□□□ C□ $AX - XB = 0$□□□□.□□□. □□ A□ B□□□□□□□, □ A□□□□□□ $f(x)$□ B□□□□□□ $g(\lambda)$□□□□□□, □□ $f(\lambda)$□ $g(\lambda)$□□, □□□□ $u(\lambda), v(\lambda)\in C[\lambda]$, □□

$$u(\lambda) f(\lambda) + v(\lambda) g(\lambda) = 1.$$

λ□ A□□, □□ Hamilton-Cayley□□, □□□ $g(A)$□□.□ C□ $AX - XB = 0$□□□□. □ $AC = CB$.□□ $g(A)C = Cg(B)$.

□ □ 9 7

1 (1) $\lambda^2 - 1$; (2) $(\lambda - 2)^3$.

2 (1) $(\lambda - 1)^2$; (2) $(\lambda - 1)^3$;

(3) $(\lambda - 3)^2(\lambda - 5)$; (4) $(\lambda - 3)^2(\lambda - 5)$.

3 □□□.

4 □ 1□□(1)□□□□□□□□□;(2)□□□□□□□□□□□.

5 □ 2□□□□ Jordan□□□□□□□□□.

6 □□: $A^m = I$, □□□□ $\lambda^m - 1$□ A□□□□□□□□.□□ $\lambda^m - 1$□□□□□□□□□, □□□□ A□□□□□□ $m(\lambda)$□□□□□.

7. □□:□□ A□□□□□□□□□, □□□□ A□□□□□□ $m(\lambda)$.□□□□ A□□□□.

8 □□:□□□□ 7□, A□□□□.

9 □□:□□□ A□□□□□□ $m(\lambda)$□ $g(\lambda)$□□, □□ $m(\lambda)$□ Q□□□□.□□ $m(\lambda)$□□□ $r>1$, □□ A□□□□□.□□□ A□□□□□, □□ $m(\lambda)$□□□□ C□□□□□

□□□□□□,□□□□□ A□□□□.

10 (1) □□:□□ A□□□□□□ $f(\lambda)$,□□ A□□□□□□ $m(\lambda)$□□□□□□ $f(\lambda)$□ K□□□□□□(□□□□□□),□□ $m(\lambda)$□□□□□□,□□□□□□,□□□□ $m(\lambda)=(\lambda-1)^2(\lambda-2)$;

(2) □□: $V=\mathrm{Ker}(A-I)^2\oplus\mathrm{Ker}(A-2I)$.

$$\alpha\in\mathrm{Ker}(A-I)^2 \Leftrightarrow (A-I)^2\alpha=0 \Leftrightarrow (A-I)^2X=0,$$

□□ X□α□□$\alpha_1,\alpha_2,\alpha_3$□□□□.□□□□□,$\alpha_1,\alpha_2-\alpha_3$□ $\mathrm{Ker}(A-I)^2$□□□□.□□□□□□ α_2□ $\mathrm{Ker}(A-2I)$□□□□.

11 □□:□□□□□□,

$$A\alpha_1=\alpha_2,\ A\alpha_2=\alpha_3,\ \dots,\ A\alpha_{n-1}=\alpha_n,$$

$$A\alpha_n=-a_0\alpha_1-a_1\alpha_2-\dots-a_{n-1}\alpha_n.$$

□□

$$A^2\alpha_1=A\alpha_2=\alpha_3,$$

$$A^3\alpha_1=A\alpha_3=\alpha_4,$$

$$\dots\dots\dots\dots$$

$$A^{n-1}\alpha_1=\alpha_n,$$

$$A^n\alpha_1=A\alpha_n,$$

$$A\alpha_n+a_{n-1}\alpha_n+\dots+a_1\alpha_2+a_0\alpha_1=0.$$

□

$$A^n\alpha_1+a_{n-1}A^{n-1}\alpha_1+\dots+a_1A\alpha_1+a_0\alpha_1=0.$$

□

$$g(\lambda)=\lambda^n+a_{n-1}\lambda^{n-1}+\dots+a_1\lambda+a_0,$$

□□□□□

$$g(A)\alpha_1=0.$$

□□ $g(A)\alpha_2=0,\dots,g(A)\alpha_n=0$.□□ $g(A)=0$.□□ $g(\lambda)$□ A□□□□□□□□.□□ $m(\lambda)\mid g(\lambda)$.□□ $\deg m(\lambda)<n$,□□□□□.□□

$$\deg m(\lambda)=n.$$

□□□□ $m(\lambda)=g(\lambda)$.□

$$m(\lambda)=\lambda^n+a_{n-1}\lambda^{n-1}+\dots+a_1\lambda+a_0.$$

□□ $m(\lambda)\mid f(\lambda)$,□ $\deg m(\lambda)=n=\deg f(\lambda)$.□□□□□ $f(\lambda)=m(\lambda)$.

*12 □□:□□□ $F[A]$□□□□□□ $g(A)$□□.□□ $g(A)\neq 0$,□□ $m(\lambda)\nmid g(\lambda)$.□□□□□□□□□□□,□□□ $g(A)$□□.

*13 □□:□ A□□□□□□□□□□$\lambda_1,\dots,\lambda_s$,□□ λ_i□□□□ n_i,$i=1,\dots,s$.□□□□□ F□□□ n□□□□□ P,□□ $P^{-1}AP=\mathrm{diag}\{\lambda_1I_{n_1},\dots,\lambda_sI_{n_s}\}$,□□ D.□ $G=P^{-1}BP$.□ $AB=BA$□□□,$GD=DG$.□□□□4□□□4 5□□18□□□□,G□□□□□□□.□ $G=\mathrm{diag}\{G_1,G_2,\dots,G_s\}$.□□ G□ $G_1,\dots,G_s$□□□□□□□□□□□,□□ G_i□□□□,$i=1,\dots,s$.

□ □ 9 8

1 (1) □□□□□ 2;

(2) □□:□□ rank $B=2$,□□ $BX=0$ □□□□□□□□□ $4-2=2$.□□ B□ Jordan □□□□ Jordan□□□□□ 2.□□□□□□□□□ B□ Jordan□□□.

2 □□: B^k □□□□□□,□□ B^k □□□□□□ $f(\lambda)=\lambda^n$.□□□□□□□ 5□§5 □□□ 2□□□.

3 □□:□□□□□.□□□.□□ B□□□□□□ 0,□□ B□□□□□□ $m(\lambda)=\lambda^l$, □□□□□□ l.

□ □ 9 9

1 (1) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$; (2) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$;

(3) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$; (4) $\begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$;

(5) $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$; (6) $\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$.

2 (1) $(\lambda-2)^2$; (2) $(\lambda-1)\lambda^2$;

(3) $(\lambda-1)^3$; (4) $(\lambda-3)(\lambda+1)^2$;

(5) $(\lambda-1)^2(\lambda+1)^2$; (6) $(\lambda-2)^2$.

3 □□:□□□□□□□:

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \cdots & \cdots & & \cdots & \cdots \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

4 □□:□□ A□ Jordan□□□□□□ 3□□□.

5 □□:□□ A□ Jordan□□□.

6 □□:□□ A□ Jordan□□□.

7. (1) □□:□□□ A□□□□□□ $f(\lambda)=\lambda^3-\lambda^2-\lambda+1$.□□□ $A^3=A^2+A-I$.□ □□□□□□□□□ $A^k=A^{k-2}+A^2-I$.

(2) □□:□□□(1)□□□□□□□□□□□□□ A^{100}:

$$A^{100}=\begin{pmatrix} 1 & 0 & 0 \\ 50 & 1 & 0 \\ 50 & 0 & 1 \end{pmatrix}.$$

□ □ 9 10

1 (1) □□; (2) □.

2 □□:□□□ $f(\alpha_1), f(\alpha_2), f(\alpha_3)$, □□□□□

$$f(x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3) = 2x_1 - 3x_2 + x_3.$$

3 □□:□ f □□□□□□□□□□□□,□□ $f(\alpha_1), f(\alpha_2), f(\alpha_3)$, □□□□□ f □□□□

$$f(x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3) = -x_1 + 5x_2 + 4x_3.$$

*4 □□:□□ f_i □ 0,□□ $\operatorname{Ker} f_i \neq V, i = 1, 2, \dots, s$. □□□□□□□ 8 2□□ 10□□□□.

5 □□:□ $(\beta_1, \beta_2, \beta_3) = (\alpha_1, \alpha_2, \alpha_3)A$, □□□□□□□□ A, □□ A □□,□□ A^{-1}. □□□□□ 1□□□□, $\beta_1, \beta_2, \beta_3$ □□□□ g_1, g_2, g_3 □□

$$(g_1, g_2, g_3) = (f_1, f_2, f_3)(A^{-1})',$$

□□

$$(A^{-1})' = \frac{1}{9}\begin{pmatrix} 2 & 1 & -2 \\ 1 & 2 & 2 \\ 2 & -2 & 1 \end{pmatrix}.$$

6 □□:□□ $V = R[x]_3$, □□ $\dim V^* = \dim V = 3$. □□ f_1, f_2, f_3 □□□□,□ f_1, f_2, f_3 □ V^* □□□□,□□ $1, x, x^2$ □ V □□□□,□□□□□□□ $\tilde{f}_1, \tilde{f}_2, \tilde{f}_3$, □□□(9)□□□□□□, $f_i = f_i(1)\tilde{f}_1 + f_i(x)\tilde{f}_2 + f_i(x^2)\tilde{f}_3, i = 1, 2, 3$. □□□□□

$$(f_1, f_2, f_3) = (\tilde{f}_1, \tilde{f}_2, \tilde{f}_3)\begin{pmatrix} 1 & 2 & -1 \\ \frac{1}{2} & 2 & \frac{1}{2} \\ \frac{1}{3} & \frac{8}{3} & -\frac{1}{3} \end{pmatrix}.$$

□□ $g_1(x), g_2(x), g_3(x)$ □□□□□ f_1, f_2, f_3, □□□□□□□ 1□

$$(g_1(x), g_2(x), g_3(x)) = (1, x, x^2)\begin{pmatrix} 1 & -\frac{1}{6} & -\frac{1}{3} \\ 1 & 0 & 1 \\ -\frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

□□□□, $g_1(x) = 1 + x - \frac{3}{2}x^2$, $g_2(x) = -\frac{1}{6} + \frac{1}{2}x^2$, $g_3(x) = -\frac{1}{3} + x - \frac{1}{2}x^2$.

7. (1) □□:□□□□□□□□□□□□□□□□;

(2) □□:□□ A^* □□□□□□□□□□;

(3) □□: $A^*(f_1, f_2, \dots, f_n) = (f_1A, f_2A, \dots, f_nA)$, □ $f_iA = \sum_{j=1}^{n} [(f_iA)(\alpha_j)] f_j = \sum_{j=1}^{n} [f_i(A\alpha_j)] f_j$. □□□

$$A(\alpha_1, \alpha_2, \dots, \alpha_n) = (\alpha_1, \alpha_2, \dots, \alpha_n)A,$$

□□□□□ $f_iA = \sum_{j=1}^{n} a_{ij} f_j$. □□□□

$$A^*(f_1, f_2, \ldots, f_n) = (f_1, f_2, \ldots, f_n) A□$$

8 (1) □□: $W = \bigcap_{i=1}^{s} \mathrm{Ker} f_i$;

*(2) □□:□□ V□□□□□ U,□ U□□□□□$\alpha_1, \alpha_2, \ldots, \alpha_m$,□□□ V□□□□ $\alpha_1, \ldots, \alpha_m, \alpha_{m+1}, \ldots, \alpha_n$,□□□□□□ $f_1, f_2, \ldots, f_n$. □□ α□ V,□□□(8)□□□□□□, $\alpha = \sum_{i=1}^{n} f_i(\alpha)\alpha_i$. □□

$$\alpha□ U \Leftrightarrow f_j(\alpha) = 0, \quad j = m+1, \ldots, n.$$

□□□□□.

□ 10 □ □□□□□□□□□

□ □ 10.1

1 (1) □□:□□□□□□□;

(2) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$;

(3) □□:□□□□□□□□;

(4) □□:□□□□□□□□;

(5) $\alpha = (1, 0, 0, 1)$□(□□□□□).

2 □□: V□□□□□$\alpha_1, \alpha_2, \ldots, \alpha_n$,□ f□□□□□□□□□ A,□ α, β□□□□□□□□□ X, Y. □□ α□ $\mathrm{rad}_L V$□□□□ X□□ $A□Z = 0$□□□□ W. □□ $\alpha \longmapsto X$ □ V□ F^n□□□□□□□,□□□□□,$\mathrm{rad}_L V$□□ W. □□

$$\dim \mathrm{rad}_L V = \dim W = n - \mathrm{rank} A□ = \dim V - \mathrm{rank}_m f.$$

□□□□ $\dim \mathrm{rad}_R V = \dim V - \mathrm{rank}_m f$.

3 (1) □□:□□□□□□□□□□□;

(2) □□:□□ α□ $\mathrm{rad}_L V$□□□□ α□ $\mathrm{Ker} L_f$;

(3) □□:□□□ 2□□□□;

(4) □□:□□□(3)□□□□□□ f□□□ ⟺ L_f□□□;□□□□ 9□§2□□□ 4.

4 □□:□□ f□□ $E_{11}, E_{12}, \ldots, E_{1n}, \ldots, E_{n1}, \ldots, E_{nn}$□□□□□.

5 (1) □□:□ V□□□□□$\alpha_1, \alpha_2, \ldots, \alpha_n$,□□ f□□□□□□□□ $f(\alpha, \beta) = x_1 y_1 + x_2 y_2 + \ldots + x_r y_r$,□□ $r = \mathrm{rank}_m f$;

(2) □□:□ $\xi = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} i, 0, \ldots, 0\right), \eta = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} i, 0, \ldots, 0\right)$.

6 □□:□□□□□□□□.

7 □□:□□□□□□□□□□□□□□.

8 (1) □□:□□ α□ $W^{\perp}$ □□□□ α □□□□□□□□□□□□□□□□□□;

(2) □□:□□□(1)□□□□.

□ □ 10 2

1 □.□□:□□□□□□□□□□□.□□□□□,□(α,α)□□□□□□.

2 □□.□□:$f(A,B)$□□□□□□□.

3 □□:□□□□ 1.

4 $\arccos\left(-\frac{1}{3}\right)$.

5 $\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{2}x, \frac{3\sqrt{10}}{4}x^2-\frac{\sqrt{10}}{4}$.

6 $\alpha_1, \frac{\sqrt{10}}{10}\alpha_2, -\frac{\sqrt{15}}{3}\alpha_1+\frac{\sqrt{15}}{15}\alpha_2+\frac{\sqrt{15}}{3}\alpha_3$.

7 □□:□□□.□□$(\beta_i,\beta_j)=\delta_{ij}$, $1\le i,j\le 3$.□□□.□□□ 8.

8 (1) $(\alpha_1,\alpha_1)=6, (\alpha_1,\alpha_2)=(\alpha_2,\alpha_1)=-2, (\alpha_1,\alpha_3)=(\alpha_3,\alpha_1)=1,$

$(\alpha_2,\alpha_2)=3, (\alpha_2,\alpha_3)=(\alpha_3,\alpha_2)=-2, (\alpha_3,\alpha_3)=3.$

(2) $\alpha_1, \frac{1}{3}\alpha_1+\alpha_2, \frac{1}{14}\alpha_1+\frac{5}{7}\alpha_2+\alpha_3$.

9 □□:□□□ V□□□□□□□□□:$\varepsilon_1, \frac{1}{2}\varepsilon_2$,□□ $\varepsilon_1=(1,0)$□, $\varepsilon_2=(0,1)$□.□□□□ $\alpha=(x_1,x_2)$□□ V□□□□□□□ $\varepsilon_1, \frac{1}{2}\varepsilon_2$ □□□□□ $(x_1, 2x_2)$□.□ α □□□□□□□□□ □σ□□ V□ R^2 □□□□□□□.

□ □ 10 3

1 $\dim U^{\perp}=2$, $U^{\perp}$ □□□□□□□

$$\beta_1=(0,-2,1,0),\quad \beta_2=\left(2,-\frac{3}{5},-\frac{6}{5},1\right).$$

□□:γ□ $U^{\perp}$ ⟺ $\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}\gamma'=0$ ⟺ γ'□ $\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}X=0$ □□.

2 $n-1$.　3 □□:□□ $U\subseteq(U^{\perp})^{\perp}$.□□□□ $\dim U=\dim(U^{\perp})^{\perp}$.

4 □□:□ $U^{\perp}$ □□□□□ $\eta_1,\eta_2,\dots,\eta_m$.□ $A=(\eta_1,\eta_2,\dots,\eta_m)$,□ U□□□□□□□ □ $A'X=0$ □□□□.

5 □□:□□ $\eta_1,\eta_2,\dots,\eta_m$ □ U□□□□□□□□,□□ $\alpha_1=\sum_{i=1}^{m}(\alpha_1,\eta_i)\eta_i$.□□□ $(\alpha_1,\eta_i)=(\alpha,\eta_i)$.

6 $\alpha_1=\left(-\frac{23}{29},-\frac{48}{29},-\frac{26}{29}\right)$.□□:□□□ 5□□□□;□□ U□□□□□□□□□ η_1,

η_2.

7. □□:□□ α_1□ U□ α□ U□□□□□□□□□□ $\alpha-\alpha_1$□ $U^{\square}$.

8 □□:□ $W=$□$\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_m$□,□ $V=W\square W^{\square}$.□□ α□ V,□ $\alpha=\alpha_1+\alpha_2$, α_1□ W, α_2□ $W^{\square}$.□□ $|\alpha_1|^2$,□□□□□□□.

9 □□:□□ $R[x]_4=W\square W^{\square}$, $\dim W=1$,□□ $\dim W^{\square}=3$.□□ W□□□□□ 1,□ □ $f(x)$□ $W^{\square}$□□□□ $(f,1)=\int_0^1 f(x)dx=0$.□□□□ $f(x)$□1□□2□□3□□□□□ □□,□□□ $W^{\square}$□□□□□

$$x-\frac{1}{2},\quad x^2-\frac{1}{3},\quad x^3-\frac{1}{4}.$$

10 □□: $\dim W=n$,□□ $\dim W^{\square}=n^2-n$.□□□, $E_{11}, E_{22},\ldots,E_{nn}$□ W□□□□□ □□□,□□ A□ $W^{\square}$□□□□ $(A, E_{ii})=0, i=1,2,\ldots,n$.□□□□□

$$W^{\square}=\{A=(a_{ij})\square M_n(R)\mid a_{11}=a_{22}=\cdots=a_{nn}=0\}.$$

□□ E_{ij}□ $W^{\square}$,□□ i□j, $i,j=1,2,\ldots,n$,□□□□□ $W^{\square}$□□□□□□□□.

□□ 10 4

1 □□:□ ξ□ A□□□□□ λ_1□□□□□,□□□□□□□ $(A\xi, A\xi)$.

2 □□:□□□□ $P\alpha=(\alpha,\eta)\eta$(□□□□ 10 3□ 5□□□□),□□ $(A\alpha, A\beta)=(\alpha,\beta)$.□□□ A□□□;□□ γ□ V,□ $\alpha=\gamma-2P\gamma$,□ $A\alpha=\gamma$.

□□□□ A□□□□□,□□η□$^{\square}$□□□□□ $\alpha_2,\ldots,\alpha_n$,□□ A□ V□□□□□$\eta,\alpha_2,\ldots,\alpha_n$□□□□,□□□□□□□□□.

3 □□: $V_1^{\square}$□1□□,□ $V_1^{\square}=$□η□. V_1□□□□□ $\alpha_1,\alpha_2,\ldots,\alpha_{n-1}$,□ $\alpha_1,\alpha_2,\ldots,\alpha_{n-1},\eta$□ V□□□□.□ P□ V□□η□□□□□□□,□

$$A\alpha_i=\alpha_i=(I-2P)\alpha_i,\quad i=1,2,\ldots,n-1.$$

□□ $$(A\eta, A\alpha_i)=(\eta,\alpha_i)=0,$$

□ $$(A\eta, A\alpha_i)=(A\eta,\alpha_i),$$

□□ $$(A\eta,\alpha_i)=0,\ i=1,2,\ldots,n-1.$$

□□ $A\eta$□ $V_1^{\square}=$□η□.□□ A□□□□□,□□ $|A\eta|=|\eta|$,□□ $A\eta=\pm\eta$.□□ $\dim V_1=n-1$,□□ $A\eta$□η.□□ $A\eta=-\eta=(I-2P)\eta$,□□ $A=I-2P$,□□ A□□□□□.

4 □□:□□□□□,□□□,□□ A□□□□:□□□□□□□ $|A(\alpha+\beta)|^2$,□□□□ $(A\alpha, A\beta)=(\alpha,\beta)$.

5 □□:□□□□ A□ V□□□□□□□□□□□□□□ A□□□□□.□ $A=(a_{ij})$. □□□□□□□□□□□□□,□□□□ $P(a_{11}, a_{21})$□□ $M(a_{12}, a_{22})$□□□□□□□□ □□□□□,□ $\overline{OP}$□ $\overline{OM}$.□□ P, M□□□□θ,φ□□□□,□ $\varphi=\theta+\frac{\pi}{2}$□ $\theta=\varphi+\frac{\pi}{2}$.□ □□□□ A□□□□□□□□.

6 □□:□ $\eta=k(\alpha-\beta)$□ $|\eta|=1$,□□ k.□□□ A□□□□□□η□$^{\square}$□□□□□,□ $A=I-2P$,□□ P□ V□□η□□□□□□□,□□ $A\alpha=\beta$.

7. □□:□ λ_0 □□□□□ A □□□□□□□□□□,□ A □□□□□,□ λ_0 □ A □□□□,□□□□□□□ α,□□ $A\alpha=\lambda_0\alpha$. □□□□□,□ α□A□$=\lambda_0\alpha$□. □□□□□□,□ α□A□$A\alpha=\lambda_0\lambda_0\alpha$□$\alpha$. □□□□ $|\lambda_0|=1$.

8 □□:□□□. □□□ 5□□□□,□□□. □□□□□□□□□□ n □□□□□□. □□□ n □□□□□□□ V □□□□□□ A□,□ A □□□□□□ λ_1,□ η_1 □ A □□□ λ_1 □□□□□□□,□ $|\eta_1|=1$. □ $V=$□η_1□ī □η_1□□. □□□η_1□□ A □□□□□□,□□□ 6□,□η_1□□ □□ A □□□□□□,□□ $A|$□η_1□□ □□η_1□□ □□□□□□□□□□□□. □□η_1□□ □□□□□,□□□□□□.

□ □ 10 5

1 $|\alpha|=3, |\beta|=2$, α □ β □□□α,β□ $=\frac{\pi}{6}$.

2 $\eta_1=\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$, $\eta_2=\left(\frac{1+i}{2}, \frac{1+i}{2}\right)$.

3 $\beta_1=(1,-1,1)$, $\beta_2=\left(\frac{2-i}{3}, \frac{1+i}{3}, \frac{-1+2i}{3}\right)$.

4 $(e^{i\theta})$, □□ θ □□□. 5 □□:□□□□□ A,□ $|\overline{A}|=|\overline{A}|$.

6 □□:□ ξ □ A □□□□□□ λ_1 □□□□□. □□□□□□□ $(A\xi, A\xi)$.

7. $\begin{pmatrix} a & b+ci \\ b-ci & d \end{pmatrix}$,□□ a, b, c, d □□□□□.

*8 □□:□ $\eta_1,\eta_2,\ldots,\eta_n$ □ V □□□□□□□□,

$$A(\eta_1,\eta_2,\ldots,\eta_n)=(\eta_1,\eta_2,\ldots,\eta_n)A,$$

□ $A\eta_i$ □□ $\eta_1,\eta_2,\ldots,\eta_n$ □□□□□□ j □□□ $a_{ji}=(A\eta_i,\eta_j)$.

*9 □□:□ ξ □ A □□□□□□ λ_1 □□□□□,□□□□□□□ $(A\xi,\xi)$.

*□ □ 10 6

1 (1) □□:□□□ f □□□□□□□□□□□□;

(2) □□:□□□□□□□□□□□;

(3) □□: $A=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$;

(4) □□:□ $\alpha=(\varepsilon_1,\varepsilon_2)X$, $\beta=(\varepsilon_1,\varepsilon_2)Y$,□ $T\alpha$, $T\beta$ □□ $\varepsilon_1,\varepsilon_2$ □□□□□□□ TX, TY. □□

$$f(T\alpha, T\beta)=(TX)□A(TY)=X□T□ATY$$

□□ $f(\alpha,\beta)=X□AY$. □□ T □□□□□□□□□

$$X□T□ATY=X□AY, \quad " x, y□ R^2.$$

□□□□□□ $T□AT=A$. □□□□□□ $T□AT=A$ □□□□ T □□□□□;

(5) □□ $\sqrt{2}+1$ □□□□□□ $k\begin{pmatrix}1\\1\end{pmatrix}$, k □□□□□□□;□□ $\sqrt{2}-1$ □□□□□□

$\begin{pmatrix} I & \\ & \begin{matrix} 1 & \\ & -1 \end{matrix} \end{pmatrix}$, I [illegible].

2 [illegible]: [illegible] $\beta = \sum_{i=1}^{n} b_i \alpha_i$, [illegible] $f(\beta, \alpha_j)$.

3 [illegible]: f [illegible] A [illegible]

$$\begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}.$$

[illegible] α, β [illegible] X, Y, [illegible] $f(\alpha, \beta) = X'AY$,

$$f(B\alpha, B\beta) = (BX)'A(BY) = X'B'ABY.$$

[illegible] B [illegible] $f(B\alpha, B\beta) = f(\alpha, \beta)$. [illegible] $B'AB = A$.

□□□□

1 □□□□□□□□□□□□□□□□□□□.□□□□.□□□.□□:□□□□□□□,1988

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